**Harold’s Business Calculus**

**Cheat Sheet**

22 December 2022

**Algebra Reference**

|  |  |  |
| --- | --- | --- |
| **Exponents** | | |
| **Multiplication** |  |  |
| **Power to a Power** |  |  |
| **Distributive** |  |  |
| **Zero Power** |  |  |
| **Power Sign Change** |  |  |
| **Negative Powers** |  |  |

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| **Radicals** | | |
| **Convert to Power** |  |  |
| **Root of a Root** |  |  |

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| **Logarithms** | | | |
| **Definition** | Log Exponential | | |
|  | | |
| **Example** |  | | |
| **Common Log** |  | *Base e assumed in pre-1955 math textbooks.*  *Base 2 assumed in computer science textbooks.* | |
| **Natural Log** |  | | *where* |
| **Powers (*x2*)** |  | |  |
| **Multiplication ()** |  | |  |
| **Division ()** |  | |  |
| **Zero (0) and One (1)** |  | |  |
| **Inverse Functions** |  | |  |
| **Change of Base** |  | | **TI-84:**  [MATH] + [A: logBASE( ] |

**3.1 Limits**

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| **Property** | (Map to Larson’s 1-pager of common derivatives) | |
| **Definition of Limit** | Let be a function and let and be real numbers. If  1. as takes values closer and closer (but not equal) to on both sides of , the corresponding values of get closer and closer (and perhaps equal) to ; and  2. the value of can be made as close to as desired by taking values of close enough to ;  then is the limit of as x approaches , written | |
| **Existence of Limits** | The limit of as approaches may not exist.  1. If becomes infinitely large in magnitude (positive or negative) as approaches the number from either side, we write  or  In either case the limit does not exist.  2. If becomes infinitely large in magnitude (positive) as approaches from one side and infinitely large in magnitude (negative) as approaches from the other side, then does not exist.  3. If and **,** and , then does not exist. | |
| **Limits at Infinity** |  |  |
| **Finding Limits at Infinity** | If , for polynomials and ,  and can be found as follows.  1. Divide and by the highest power of in .  2. Use the rules for limits, including the rules for limits at infinity,  and  to find the limit of the result from Step 1. | |

**Rules for Limits**

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| **Rule** | (Map to Larson’s 1-pager of common derivatives) |
| **Given** | Let , , and be real numbers, and let and be functions such that    and  . |
| **1. Constant (*c*)** | If is a constant, then  and |
| **2. Sum or Difference (+, )** | The limit of a sum or difference is the sum or difference of the limits. |
| **3. Product ()** | The limit of products is the product of the limits. |
| **4. Quotient ()** | The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.  if . |
| **5. Polynomial ()** | If is a polynomial, then |
| **6. Exponent (*xk*)** | For any real number ,  provided that this limit exists. |
| **7. Equivalent Functions (=)** | If for all . |
| **8. Function Exponent** | For any real number , |
| **9. Logorithm** | For any real number such that or ,  if . |

**3.2 Continuity**

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| **Term** | **Definition** |
| **Continuity at** | A function is **continuous** at if the following three condistions are satisfied:  1. is defined,  exists, and  3.  If is not continuous at , it is **discontinuous** there. |
|  |
| **Continuity on a Closed Interval** | A function is **continuous on a closed interval** if  1. It is continuous on the open interval ,  2. It is continuous from the right at , and  3. It is continuous from the left at . |

**3.3 Rates of Change**

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| **Term** | **Equation** |
| **Average Rate of Change** | The **average rate of change** of with respect to for a function as changes from to is |
| **Instantaneous Rate of Change** | The **instantaneous rate of change** for a function when is  or  provided this limit exists. |

**3.4 Definition of the Derivative**

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| **Term** | **Definition** | | |
| **Slope of the Tangent Line** | The **tangent line** of the graph of at the point is the line through this point having slope  provided this limit exists.  If the limit does not exist, then there is no tangent at that point. | | |
| **Derivative** | The **derivative** of the function at is defined as  or  provided this limit exists.  The derivative is the **slope generating function** at any point *x*.  It is usually set = 0 to find minimum (loss) and maximum (profit) values of . | | |
|  |
| **Notations for the Derivative of** |  | |  |
| **Equivalent Expressions for the Change in** |  | Useful for describing the equation of a line through two points. | |
|  | A way to write without the subscripts. | |
| *x* | Useful for describing the change in without referring to the individual points. | |
| *h* | A way to write *x* with just one symbol. | |
| **Equation of the Tangent Line** | The **tangent line** to the graph of at the point is given by the equation  provided exists. | | |
| **Existence of the Derivative** | The derivative **exists** when a function satisfies *all* of the following conditions at a point.  1. is continuous,  2. is smooth, and  3. does not have a vertical tangent line.  The derivative does **not exist** when *any* of the following conditions are true for a function at a point.  1. is discontinuous,  2. has a sharp corner, or  3. has a vertical tangent line. | | |
|  |

**4. Derivative Formulas**

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| **Rule** | **Formula** |
| **1. Chain Rule (**See the source image**)** |  |
| **2. Constant Rule (*c*)** |  |
| **3. Constant Multiple Rule (*c*)** |  |
| **4. Sum and Difference Rule (+, )** |  |
| **5. Product Rule ()** |  |
| **6. Quotient Rule ()** |  |
| **7. Power Rule (*xn*)** |  |
| **8. General Power Rule (*xn*)** |  |
| **9. Power Rule for** |  |
| **10. Natural Exponential Rule** |  |
| **11. General Natural Exponential Rule** |  |
| **12. Exponential Rule** |  |
| **13. General Exponential Rule** |  |
| **14. Natural Logorithm Rule** |  |
| **15. General Natural Logorithm Rule** |  |
| **16. Logorithm Rule** |  |
| **17. General Logorithm Rule** |  |

**Equation of a Line**

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| **Form** | **Equation** |
| **Genreal Form** |  |
| **Slope-Intercept Form** |  |
| **Point-Slope Form** |  |
| **Calculus Form** |  |
| **Slope** |  |

**4.1 Derivative Applications**

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| **Application** | **Business Case** |
| **Average Cost** | If the total cost to manufacture items is given by , then the **average cost** per item is . |
| **Marginal Average Cost** | The **marginal average cost** is the derivative of the average cost function, . |
|  |  |
| **Profit** | Profit equals total Revenue minus Cost or Expenses. |
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**5. Graphing with Derivatives**

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| **Term** | **Definition** |
| **Test for Increasing and Decreasing Functions** | 1. If , then is increasing (slope up). ↗  2. If , then is decreasing (slope down). ↘  3. If , then is constant (zero slope). → |
| **Critical Numbers** | The **critical numbers** for a function are those numbers in the domain of for which or does not exist. |
| **Critical Point** | A **critical point** is a point whose -coordinate is the critical number and whose -coordinate is. |
| **First Derivative**  (Slope Formula) | *finds critical points (min. and max.).*  *Don’t forget to check the boundaries: and .* |
| **First Derivative Test** | 1. If changes from – to + at *c*, then has a *relative minimum* at *.*  2. If changes from + to - at *c*, then has a *relative maximum* at *.*  3. If , is + *c* + or − *c* −, then is neither. |
| **Test for Concavity** | 1. If for all , then the graph is concave up. ⋃  2. If for all , then the graph is concave down. ⋂ |
| **Second Deriviative Test**  Let *f ’(c)*=0, and *f ”(x)* exists, then | 1. If , then has a relative minimum at .  2. If , then has a relative maximum at .  3. If , then the test fails (See derivative test).  If , then cup up (min.)  If, then cup down (max.) |
| **Points of Inflection**  Change in concavity | If is a point of inflection of , then either  1. or  2. does not exist at *x = c.* |
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**5.4 Curve Sketching**

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| **Step** | **Description** | |
| To sketch the graph of a function : | | |
| **1.** | Consider the domain of the function, and note any restrictions.  (That is, avoid dividing by 0, taking a square root, or any even root, of a negative number, or taking the logarithm of 0 or a negative number.) | |
| **2.** | Find the -intercept (if it exists) by substituting into .  Find any -intercepts by solving if this is not too difficult. | |
| **3.** | (a) If is a rational function, find any vertical asymptotes (VA) by investigating where the denominator is 0, and find any horizontal asymptotes (HA) by finding the limits as and .  (b) If is an exponential function, find any horizontal asymptotes (HA);  If is a logarithmic function, find any vertical asymptotes (VA). | |
| **4.** | Investigate symmetry.  If , the function is **even**, so the graph is symmetric about the *y*-axis.  If ,, the function is **odd**, so the graph is symmetric about the origin. | |
| **5.** | Find .  Locate any critical points by solving the equation and determining where does not exist, but does.  Find any relative extrema and determine where is increasing or decreasing. | |
| **6.** | Find .  Locate potential inflection points by solving the equation and determining where does not exist.  Determine where is concave upward or concave downward. | |
| **7.** | Plot the and intercepts, the critical points, the inflection points, the asymptotes, and other points as needed.  Take advantage of any symmetry found in Step 4. | |
| **8.** | Connect the points with a smooth curve using the correct concavity, being careful not to connect points where the function is not defined. | |
| **9.** | Check your graph using a graphing calculator or [desmos](https://www.desmos.com/).  If the picture looks very different from what you’ve drawn, see in what ways the picture differs and use that information to help find your mistakes. | |
| **Example Chart** | |  |

**6.1 Absolute Extrema**

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| **Term** | **Definition** |
|  | |
| **Absolute Maximum** | Let *ƒ* be a function defined on some interval.  Let be a number in the interval.  Then is the **absolute maximum** of on the interval if  for every in the interval. |
| **Absolute Minimum** | Let *ƒ* be a function defined on some interval.  Let be a number in the interval.  Then is the **absolute minimum** of on the interval if  for every in the interval. |
| **Absolute Extremum (Extrima)** | A function has an **absolute extremum** (plural: **extrema**) at if it has either an absolute maximum or an absolute minimum there. |
| **Extreme Value Theorem** | A function that is continuous on a closed interval [*a, b*] will have both an absolute maximum and an absolute minimum on the interval. |
| **Finding Absolute Extrema** | To find absolute extrema for a function continuous on a closed interval [*a, b*]:  1. Find all critical numbers for in (*a, b*).  2. Evaluate for all critical numbers in (*a, b*).  3. Evaluate for the *endpoints* *a* and *b* of the interval [*a, b*].  4. The largest value found in Step 2 or 3 is the absolute maximum for on [*a, b*], and the smallest value found is the absolute minimum for on [*a, b*]. |
| **Critical Point Theorem** | Suppose a function is continuous on an interval *I* and that has exactly one critical number in the interval *I*, located at .  If has a relative maximum at , then this relative maximum is the absolute maximum of on the interval *I*.  If has a relative minimum at , then this relative minimum is the absolute minimum of on the interval I. |

**6.2 Applications of Extrema**

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| **Step** | **Description** |
| **Solving an Applied Extrema Problem** | |
| **1.** | Read the problem carefully.  Make sure you understand what is given and what is unknown. |
| **2.** | If possible, sketch a diagram.  Label the various parts. |
| **3.** | Decide which variable must be maximized or minimized.  Express that variable as a function of one other variable. |
| **4.** | Find the domain of the function. |
| **5.** | Find the critical points for the function from Step 3. |
| **6.** | If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute maximum or minimum.  If the domain is an open interval, apply the critical point theorem when there is only one critical number.  If there is more than one critical number, evaluate the function at the critical numbers and find the limit as the endpoints of the interval are approached to determine if an absolute maximum or minimum exists at one of the critical points. |

**6.3 Further Business Application**

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| **Economic Lot Size, Economic Order Quantity, Elasticity of Demand** | |
| **Elasticity of Demand** | Let , where is demand at a price . The **elasticity of demand** is  *Demand is* ***inelastic*** *if .*  *Demand is* ***elastic*** *if .*  *Demand has* ***unit******elasticity*** *if .* |
|  |  |
| **Revenue and Elasticity** | 1. If the demand is inelastic, total revenue increases as price increases.  2. If the demand is elastic, total revenue decreases as price increases.  3. Total revenue is maximized at the price where demand has unit elasticity. |

**6.4 Implicit Differentiation**

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| **Term** | **Definition** | |
| **Implicit Differentiation** | To find for an equation containing and :  1. Differentiate on both sides of the equation with respect to , keeping in mind that is assumed to be a function of.  2. Using algebra, place all terms with on one side of the equals sign and all terms without on the other side.  3. Factor out , and then divide to solve for . | |
| **Example** |  | Product Rule  Chain Rule  Distribute.  Subtract  Divide  Cancel common term ()  Bring powers to numerator  Simplify  Cleanup negative and unit exponents |

**6.5 Related Rates**

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| **Term** | **Definition** |
| **Solving a Related Rates Problem** | 1. Identify all given quantities, as well as the quantities to be found. Draw a sketch when possible.  2. Write an equation relating the variables of the problem.  3. Use implicit differentiation to find the derivative of both sides of the equation in Step 2 with respect to time ().  4. Solve for the derivative giving the unknown rate of change and substitute the given values. |
| **Example** | Steps to solve:   1. Identify the known variables and rates of change. 2. Construct an equation relating these quantities. 3. Differentiate both sides of the equation. 4. Solve for the desired rate of change. 5. Substitute the known rates of change and quantities into the equation. |
| http://i1104.photobucket.com/albums/h330/mathclassroom/Calculus/ladder.png |

**6.6 Differentials: Linear Approximation**

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| **Differentials** | **Formula** |
| **Differentials** | For a function whose derivative exists, the differential of , written , is an arbitrary real number (usually small compared with ); the differential of , written , is the product of and , or  or |
| **Relative Error** | Relative Error in % |
| **Linear Approximation** | Let be a function whose derivative exists.  For small nonzero values of ,  and  or |
|  |  |
| **Example** | Solve for  Rewrite as |

**7.1 Antiderivative / Integration**

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| **Rule** | **Formulas** |
| **Antiderivative** | If , then is an antiderivative of . |
| **Notation** |  |
| **Concept** | If and are both antiderivatives of a function on an interval, then there is a constant *C* such that  (Two antiderivatives of a function can differ only by a constant.)  The arbitrary real number *C* is called an integration constant. |
| **Indefinite Integral** | If , then  for any real number *C*. |
| **Power Rule ()** | For any real number , |
| **Constant Multiple Rule ()** |  |
| **Sum or Difference Rule (+, )** |  |
| **Indefinite Integrals of Exponential Functions ()** | For *:* |
| **Natural Exponential Rule ()** |  |
| **Indefinite Integral of** |  |

**7.2 Integral Substitution**

|  |  |  |  |
| --- | --- | --- | --- |
| **Method** | **Formula** | | |
| **Substitution** | Each of the following forms can be integrated using the substitution . | | |
|  | **Form of the Integral** | **Result** |
| 1. |  |  |
| 2. |  |  |
| 3. |  |  |
| **Substitution Method** | In general there are three cases.  We **choose**  to be one of the following:  1. the quantity under a root or raised to a power;  2. the quantity in the denominator;  3. the exponent on .  Always capture the constant in , such as .  Remember that some integrands may need to be rearranged to fit one of these cases. | | |
| *\_\_* | | |

**7.3 Area and the Definite Integral**

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| **Definition** | **Formula** |
| **The Definite Integral** | If is defined on the interval [*,* ], the definite integral of from to is given by  provided the limit exists, where  and is any value of in the *i*th interval.  aka Riemann sum. |
| **Total Change in** | If gives the rate of change of for in [*,* ], then the total change in as goes from to is given by |

**7.4 The Fundamental Theorem of Calculus**

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| **Definition** | **Formula** |
| **Fundamental Theorem of Calculus** | Let be continuous on the interval [*a, b*], and let be any antiderivative of . Then |
| **1. Constant Multiple of a Function ()** |  |
| **2. Sum or Difference of Functions (+, )** |  |
| **3. Same Bounds** |  |
| **4. Split Bounds** |  |
| **5. Swap Bounds** |  |
| **Finding Area** | In summary, to find the area bounded by , and the -axis, use the following steps.  1. Sketch a graph.  2. Find any -intercepts of in [*a, b*].  These divide the total region into subregions.  3. The definite integral will be positive for subregions above the -axis and negative for subregions below the -axis.  Use separate integrals to find the (positive) areas of the subregions.  4. The total area is the sum of the areas of all of the subregions. |
|  |

**7.5 The Area Between Two Curves**

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| **Definition** | **Formula** |
|  | |
| **Area Between Two Curves** | If and are continuous functions and on [*a, b*], then the area between the curves and from to is given by |
| **Consumers’ Surplus** | If is a demand function with equilibrium price and equilibrium demand , then **Customers’ Surplus** is given by |
|  |  |
| **Producers’ Surplus** | If is a supply function with equilibrium price and equilibrium supply , then **Producer’s Surplus** is given by |
|  |  |

**7.6 Numerical Integration**

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| **Rule** | | **Formula** |
| **Trapezoidal Rule** | | Let be a continuous function on [*a, b*] and let [*a, b*] be divided into *n* equal subintervals by the points *a = x0, x1, x2, ... , xn = b*.  Then, by the **trapezoidal rule**,  and |
|  | |
| **Simpson’s Rule** | | Let be a continuous function on [*a, b*] and let [*a, b*] be divided into *n* equal subintervals by the points *a = x0, x1, x2, ... , xn = b*.  Then by **Simpson’s rule**,  Where is even and |
|  | |
| **TI-84** | http://www.gosale.com/product_images/4948000/texas-instruments-ti-84-plus-4948122.jpg | [MATH] fnInt(f(x),x,a,b), [MATH] [1] [ENTER]  Example: [MATH] fnInt(x^2,x,0,1) |
| **TI-Nspire CAS** | http://i5.walmartimages.com/dfw/dce07b8c-6b5e/k2-_10014b04-7e9e-4157-a2ec-ab5b03fa1234.v1.jpg | [MENU] [4] Calculus [3] Integral  [TAB] [TAB]  [X] [^] [2] [TAB]  [TAB] [X] [ENTER]  Shortcut: [ALPHA] [WINDOWS] [4] |

**Source**

* All highlighted formulas copied from chapters 3−7 of “[Calculus with Applications](https://www.readallbooks.org/book/calculus-with-applications-global-edition/#download)”, 11th Edition (Global Edition), by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey, Pearson, 2017.
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