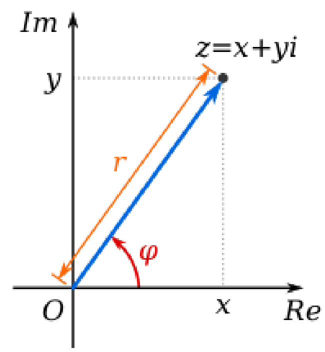
Harold's Complex Variables
Cheat Sheet
23 April 2024

Definitions

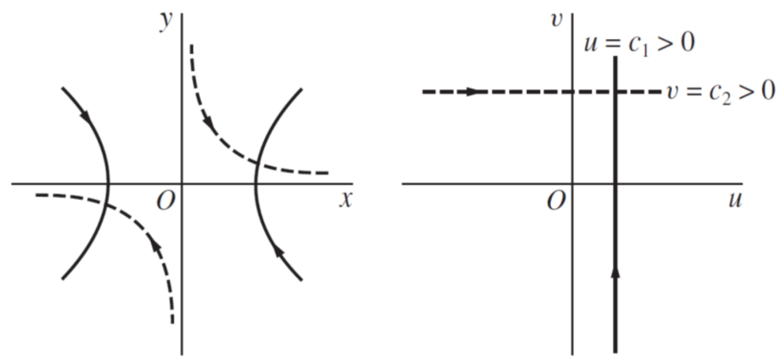
Term	Formulas
Imaginary Number	$i = \sqrt{-1}$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$ i is used by mathematicians. j is used by engineers.
Complex Number	 Rectangular Form (x, y): $z = x + iy$ $z = (x, y)$ where $x = \text{Re}\, z;\ y = \text{Im}\, z$ Polar Form (r, θ): $z = r(\cos\theta + i\sin\theta)$ Exponential Form (e^x): $z = re^{i\theta}$ Parametric Form (ρ, θ): $z = z_0 + \rho e^{i\theta}$ $(0 \le \theta \le 2\pi)$ Shorthand: $e^z = \exp(z) = e^x e^{iy}$
Complex Conjugate	$\bar{z} = x - iy$ $z = r(\cos\theta - i\sin\theta)$ $z = re^{-i\theta}$
Modulus (Magnitude/Absolute Value)	$\|z\| = \sqrt{x^2 + y^2}$ $\|z\| = r$ $\|z\|^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2 = z\bar{z}$
Argument (Angle)	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ If $(-\pi < \theta \le \pi)$ then principle value

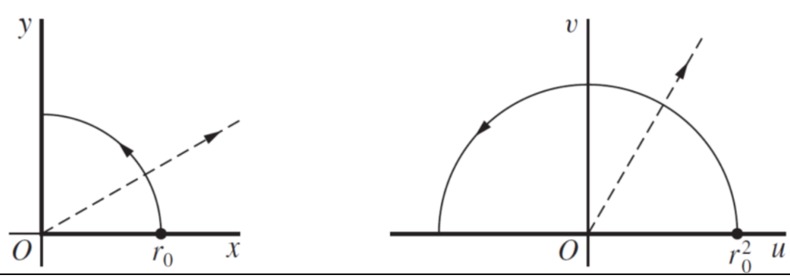
Euler's Formula	$e^{i\theta} = \cos\theta + i\sin\theta$ Examples: $e^{i\frac{\pi}{2}} = i$ $e^{i\pi} = -1$ $e^{-i\frac{\pi}{2}} = -i$ $e^{i2\pi} = 1$
De Moivre's Formula	$z^n = [r\,(\cos\theta + i\sin\theta)]^n$ $= r^n(\cos n\theta + i\sin n\theta)$
Reflection Principle	$\overline{f(z)} = f(\bar{z})$ If the lower half is the reflection of the upper half with respect to the x-axis.

Algebraic Properties

Property	Formulas
Complex Numbers	$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ $z_1 - z_2 = (x_1 + x_2) - i(y_1 + y_2)$ $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ $\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}, z_2 \neq 0$ $\lvert z_1 \cdot z_2 \cdot z_3 \dots z_n \rvert = \lvert z_1 \rvert \cdot \lvert z_2 \rvert \cdot \lvert z_3 \rvert \dots \lvert z_n \rvert$ $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
Additive Inverses	$-z = (-x, -y)$ $-z = re^{i(\theta+\pi)}$
Multiplicative Inverses	$z^{-1} = \left(\frac{x}{x^2 + y^2}, i\frac{-y}{x^2 + y^2}\right), z \neq 0$ $z^{-1} = \frac{1}{r} e^{-i\theta}$
Complex Conjugates	$\lvert \bar{z} \rvert = \lvert z \rvert$ $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$
Triangle Inequality	$\lvert z_1 \pm z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$ $\lvert z_1 \pm z_2 \rvert \geq \lvert z_1 \rvert - \lvert z_2 \rvert$ $\lvert z_1 + z_2 \rvert \geq \big\lvert \lvert z_1 \rvert - \lvert z_2 \rvert \big\rvert$
Exponentials	$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$ $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$ $z^n = r^n e^{in\theta}$ $e^z = e^{z + 2\pi i}$
Roots	$\sqrt[n]{z} = \sqrt[n]{r} \exp\left[i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$
Arguments (Angles)	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ $\arg(z_2^{-1}) = -\arg(z_2)$

Functions

<table>
<tr><th>Formula Name</th><th>Formulas</th></tr>
<tr><td>Functions</td><td>$f(z) = f(x + iy) = u(x, y) + iv(x, y) = u + iv$
$f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta) = u + iv$</td></tr>
<tr><td>Conic Mappings</td><td>Hyperbola (Rectangular Form):
$w = z^2$
$u = x^2 + y^2 = c_1$
$v = 2xy = -2y\sqrt{y^2 + c_1}$


Circle (Polar Form):
$w = z^2$
$w = r^2 e^{i2\theta}$
$\rho = r^2$
$\varphi = 2\theta$

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</table>

Transcendental Properties

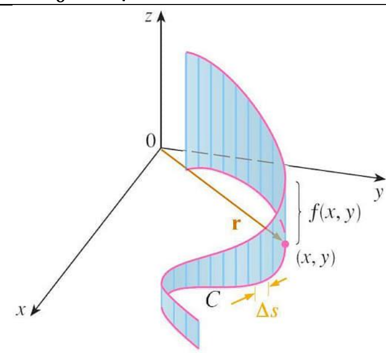
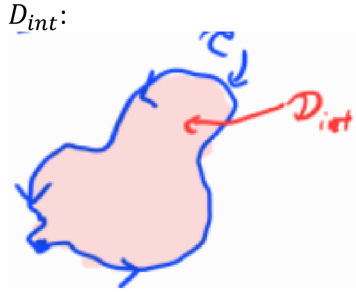
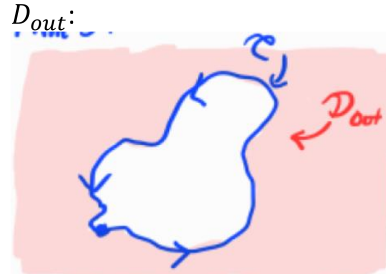
Property	Formulas
Logarithms	$\log z = \ln\lvert z\rvert + i\arg z$ $\log e^z = z + 2n\pi i$ $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ $\ln\lvert z_1 z_2\rvert = \ln\lvert z_1\rvert + \ln\lvert z_2\rvert$ $\ln\lvert z_1 z_2\rvert + i\arg(z_1 z_2) = (\ln\lvert z_1\rvert + i\arg(z_1)) + (\ln\lvert z_2\rvert + i\arg(z_2))$
Power	$z^c = e^{c\log z} = \exp(c\log z)$
Trigonometric	$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ $\sin^2 z + \cos^2 z = 1$ $\tan^2 z + 1 = \sec^2 z$ $1 + \cot^2 z = \csc^2 z$ $\sin(2z) = 2\sin z\cos z$ $\cos(2z) = \cos^2 z - \sin^2 z$ $\cos(2z) = 2\cos^2 z - 1$ $\cos(2z) = 1 - 2\sin^2 z$
Hyperbolic	$\sinh z = \frac{e^z - e^{-z}}{2}$ $\cosh z = \frac{e^z + e^{-z}}{2}$ $\sin(ix) = i\sinh x$ $\cos(ix) = \cosh x$ $\sin z = \sin x\cosh y + i\cos x\sinh y$ $\cos z = \cos x\cosh y - i\sin x\sinh y$ $\sinh z = \sinh x\cos y + i\cosh x\sin y$ $\cosh z = \cosh x\cos y - i\sinh x\sin y$
Inverse Trigonometric	$\sin^{-1} z = -i\ln\left[iz + \sqrt{1 - z^2}\right]$ $\cos^{-1} z = -i\ln\left[z + \sqrt{1 - z^2}\right]$ $\tan^{-1} z = \frac{i}{2}\ln\left[\frac{i + z}{i - z}\right]$ $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\tan^{-1} z = \frac{1}{1 + z^2}$

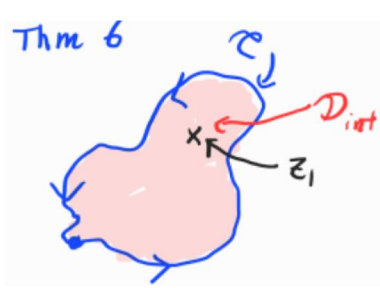
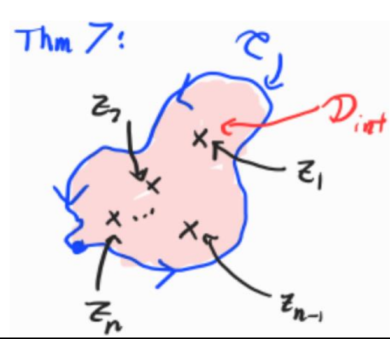
Inverse Hyperbolic	$\sinh^{-1} z = \ln\left[z + \sqrt{1+z^2}\right]$ $\cosh^{-1} z = \ln\left[z + \sqrt{1+z^2}\right]$ $\tanh^{-1} z = \frac{1}{2}\ln\left[\frac{1+z}{1-z}\right]$

Differentiation

Formula Name	Formulas
Cauchy-Riemann Equations	Determines whether the given complex valued function $f(z) = u + iv$ is analytic and differentiable. Rectangular Form: $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists at point $z_0 = x_0 + iy_0$ $\boldsymbol{u_x = v_y}, \quad \boldsymbol{u_y = -v_x}$ $f'(z_0) = u_x + iv_x$ where $u_x = \frac{\partial u}{\partial x}$ Polar Form: $f(z) = u(r, \theta) + iv(r, \theta)$ $ru_r = v_\theta, \quad u_\theta = -rv_r$ $f'(z_0) = e^{-i\theta}(u_r + iv_r)$
Laplace's Equation (Harmonic)	$H_{xx}(x, y) + H_{yy}(x, y) = 0$

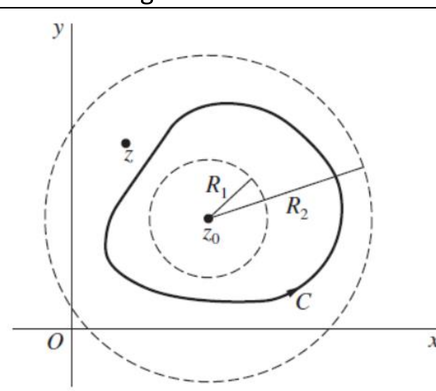
Integration

Formula Name	Formulas
Fundamental Theorem of Calculus with Contour Integral	$$\oint_C f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$ where z_0 is a point within the contour C.
Line Integral (Real) = Contour Integral (Complex)	
Contour (C)	A closed path in the complex plane.
Simple Arc (C) (Jordan arc)	If arc C does not cross itself; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. E.g., open.
Simple Curve (C)	A simple arc where $z(b) = z(a)$. E.g., closed.
Positively Oriented	a *simple closed curve,* or a Jordan curve is positively oriented when it is in the counterclockwise direction.
Branch Cut	A portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f.
Contour Integral	$$\oint_C \boldsymbol{f(z)\,dz} = \int_a^b \boldsymbol{f[z(t)]\,z'(t)\,dt}$$ $$\oint_{-C} f(z)\,dz = -\oint_C f(z)\,dz$$
Regions Bound by Curve C	D_{int}:  D_{out}: 

<table>
<tr><td>Cauchy-Goursat Theorem
(Cauchy's Integral Theorem)</td><td colspan="2">D_{int}: If C is closed, i.e., $z_0 = z_1$, then
$$\oint_C f(z)\,dz = 0$$
D_{out}: Outside of closed C, at infinity (∞):
If
$$\lim_{z\to\infty} f(z)\,z = 0$$
Then
$$\oint_C f(z)\,dz = 0$$</td></tr>
<tr><td>Cauchy Integral Formula</td><td colspan="2">Simple:
$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$
Extension:
$$\boldsymbol{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz}$$
where $n = 0, 1, 2, \ldots;\ 0! = 1;\ f^{(0)}(z_0) = f(z_0)$.
NOTE: Turns a cyclic integral into a derivative.</td></tr>
<tr><td>Jordan's Lemma</td><td colspan="2">Estimation Lemma:
$$\left|\int_C f(z)\,dz\right| \le \text{length}(C) \cdot \max_{z\in C}|f(z)|$$
Common Application:
If
$$\lim_{z\to\infty} f(z)\,z = 0$$
Then
$$\int_{-\infty}^{\infty} f(at + b)\,a\,dt\,x = \int_L f(z)\,dz = \lim_{R\to\infty}\oint_{C_R} f(z)\,dz$$
where line $L = \{at + b\}, -\infty < t < \infty$ and
R = semi-circle radius along this line.</td></tr>
<tr><td>Closed, Simple, Counter-Clockwise Oriented Curve</td><td>One Point, Simple Pole:

</td><td>Multiple Points, Simple Poles:

</td></tr>
<tr><td>Pole</td><td colspan="2">Roots in the denominator of a complex function that is homomorphic (complex differential). E.g., Singularity, asymptote.</td></tr>
</table>

Cauchy's Residue Theorem	<u>One</u> Point, Simple Pole inside Contour C: If exists $$\mathbf{Res}_{z_1}(f) = \lim_{z\to z_1}(z - z_1)\, f(z)$$ Then $$\oint_C f(z)\, dz = 2\pi i \,(\mathrm{Res}_{z_1}(f))$$ <u>Multiple</u> Points, Simple Poles inside Contour C: If these exist $$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) =$$ $$Res_{z_1}(f) = \lim_{z\to z_1}(z - z_1)\, f(z)$$ $$\ldots$$ $$Res_{z_n}(f) = \lim_{z\to z_n}(z - z_n)\, f(z)$$ Then $$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$ Higher-Order Poles: $$R_n = \frac{1}{(n-1)!} \lim_{z\to p}\left[\frac{d^{n-1}}{d_z^{n-1}}\{(z-p)^n\, f(z)\}\right]$$ where p is a pole in the contour region.

Series

Formula Name	Formulas
Liouville's Theorem	If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.
Fundamental Theorem of Algebra	$$P(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0, \qquad (a_n \neq 0, n \geq 1)$$ $$P(z) = c(z - z_n) \ldots (z - z_2)(z - z_1)$$ Any polynomial of degree n has at least one zero in the complex plane. That is, there exists at least one point z_0 such that $P(z_0) = 0$.
Maximum Modulus Principle	**Theorem:** If a function f is analytic and not constant in a given domain D, then the modulus $\lvert f(z)\rvert$ has no maximum value in D. That is, there is no point z_0 in the domain such that $\lvert f(z)\rvert \leq \lvert f(z_0)\rvert$ for all points z in it. **Corollary**: Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $\lvert f(z)\rvert$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.
Complex Variable Convergence	$$\lim_{n\to\infty} z_n = z$$ $$\text{iff } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$ $$\text{where } z_n = x_n + iy_n$$
Complex Series Convergence	$$\sum_{n=1}^{\infty} z_n = S$$ $$\text{iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$$ $$\text{where } S = X + iY$$
Series Convergence	**Corollary 1**: If a series of complex numbers converges, the n[th] term converges to zero as n tends to infinity. **Corollary 2:** The absolute convergence of a series of complex numbers implies the convergence of that series.
Annular Domain $R_1 < \lvert z - z_0\rvert < R_2$	
Transcendental Series	See Harold's Taylor Series Cheat Sheet for a comprehensive list of the Taylor series of all transcendental functions.

Taylor Series	$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where disk } (\lvert z - z_0 \rvert < R_0)$$ $$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{where } (n = 0, 1, 2, \dots)$$ Series converges to $f(z)$ when z lies in the stated open disk. If $z_0 = 0$, then **Maclaurin series**.
Laurent Series	General Form: $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$ $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \text{ where } (n = 0, 1, 2, \dots)$$ $$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad \text{where } (n = 1, 2, \dots)$$ $$\text{where } (R_1 < \lvert z - z_0 \rvert < R_2)$$ Taylor Series Form: $$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$ $$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!} \quad \text{where } (n = 0, 1, 2, \dots)$$ $$\text{and } (R_1 < \lvert z - z_0 \rvert < R_2)$$ If no poles, then **Taylor series**.

Power Series

<table>
<tr><th>Formula Name</th><th>Formulas</th></tr>
<tr><td rowspan="2">Absolute and Uniform Convergence</td><td>Theorem 1: If a power series
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$
converges when
$$z = z_1 \ (z_1 \neq z_0),$$
then it is absolutely convergent at each point z in the open disk
$$|z - z_0| < \ R_1 \text{where } R_1 = |z_1 - z_0|.$$</td></tr>
<tr><td>Theorem 2: If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$
then that series must be uniformly convergent in the closed disk
$$|z - z_0| \leq \ R_1 \text{where } R_1 = |z_1 - z_0|.$$</td></tr>
<tr><td>Continuity of Sums</td><td>Theorem: A power series
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$
represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z0| = R$.</td></tr>
<tr><td>Integration</td><td>Theorem: Let C denote any contour interior to the circle of convergence of the power series
$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$
and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is,
$$\int_C g(z)\, S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)\, (z - z_0)^n dz$$</td></tr>
<tr><td>Differentiation</td><td>Theorem: The power series
$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$
can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,
$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$</td></tr>
<tr><td>Leibniz's Rule for the n[th] Derivative</td><td>$$[f(z)\, g(z)]^{(n)} = \sum_{n=0}^{\infty} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$$
$$where\ (n = 1, 2, \dots)$$
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \quad \text{where } (k = 0, 1, 2, \dots, n)$$</td></tr>
</table>

<table>
<tr><td rowspan="2">Uniqueness Representations</td><td colspan="2">Theorem 1: If a series $$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$ converges to $f(z)$ at all points interior to some circle $|z-z_0| = R$, then it is the Taylor series expansion for f in powers of $z-z_0$.</td></tr>
<tr><td colspan="2">Theorem 2: If a series $$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z-z_0$ for that domain.</td></tr>
<tr><td>Let ...</td><td>$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$ $$g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$</td><td>$$h(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$$ $$i(z) = \sum_{n=0}^{\infty} d_n(z-z_0)^n$$ where $(|z-z_0| < R)$</td></tr>
<tr><td>Multiplication</td><td colspan="2">$f(z)g(z) = h(z)$</td></tr>
<tr><td>Division</td><td colspan="2">$$\frac{f(z)}{g(z)} = i(z)$$</td></tr>
</table>

Sources

- For NYU MATH-UY-4434 – Applied Complex Variables, Complex Variables and Applications, 9th Edition, Chapters 1-7, James Ward Brown & Ruel V. Churchill, McGraw-Hill Education, 2014.