Harold's Logic Cheat Sheet

4 November 2024

The Seven Basic Logical Symbols

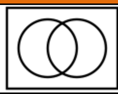
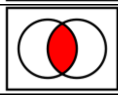
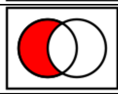
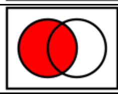
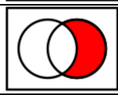
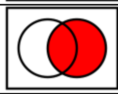
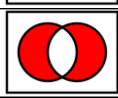
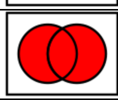
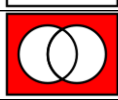
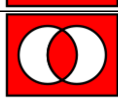
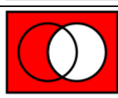
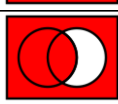
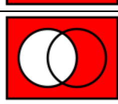
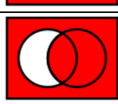
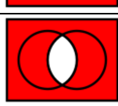
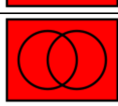
Operator	Symbol	Example	English
1) Intersection	∧, **∧**, ∧, ⋀, ⋀	p ∧ q	• Conjunction • p and q • p, but q • despite the fact that p, q • even though p, q • although p, q • overlap
2) Union	∨, **∨**, ∨, ⋁, ⋁	p ∨ q	• Disjunction • p or q • inclusive or • both combined
3) Negation	¬, ¬	¬p	• not p
4) Conditional	→, →, →, ⟶, ⇒, ⟹	p → q	• if p then q • if p, q • q if p • p implies q • p only if q • q in case that p • p is sufficient for q • q is necessary for p
5) Biconditional	↔, ↔, **↔**, ⇔, ⟺	p ↔ q	• p iff q • p if and only if q • p is necessary and sufficient for q • if p then q, and conversely • if not p then not q, and conversely
6) Universal Quantifier	∀*x*	∀*x p(x)*	• for all • for any • for each
7) Existential Quantifier	∃*x*	∃*x p(x)*	• there exists • there is at least one
Equivalence	≡, ≡, ≡	$expression_1 \equiv expression_2$	• is identical to • is equivalent to • is defined as • the two expressions always have the same truth value

- The structure of all mathematical statements can be understood using these symbols.
- All mathematical reasoning can be analyzed in terms of the proper use of these symbols.

Logical Connective Laws / Equivalences

Law	Union Example	Intersection Example
Identity Laws	$p \lor F \equiv p$	$p \land T \equiv p$
Domination, Null, or Universal Bound Laws	$p \lor T \equiv T$	$p \land F \equiv F$
Idempotent Laws	$p \lor p \equiv p$	$p \land p \equiv p$
Double Negations or Involution Law	$\neg\neg p \equiv p$	
Negation, Complement, or Complementary Laws	$p \lor \neg p \equiv T$ $\neg F \equiv T$	$p \land \neg p \equiv F$ $\neg T \equiv F$
Commutative Laws	$p \lor q \equiv q \lor p$	$p \land q \equiv q \land p$
Associative Laws	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$(p \land q) \land r \equiv p \land (q \land r)$
Distributive Laws	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
Uniting Laws	$(p \land q) \lor (p \land \neg q) \equiv p$	$(p \lor q) \land (p \lor \neg q) \equiv p$
Absorption Laws	$p \lor (p \land q) \equiv p$	$p \land (p \lor q) \equiv p$
De Morgan's Law (Propositional Logic)	$p \lor q \equiv \neg(\neg p \land \neg q)$ $\neg(p \lor q) \equiv \neg p \land \neg q$ $(p \lor \neg q) \rightarrow r \equiv \neg r \rightarrow (\neg p \land q)$	$p \land q \equiv \neg(\neg p \lor \neg q)$ $\neg(p \land q) \equiv \neg p \lor \neg q$
Multiplying and Factoring Laws	$(p \lor q) \land (\neg p \lor r) \equiv$ $(p \land r) \lor (\neg p \land q)$	$(p \land q) \lor (\neg p \land r) \equiv$ $(p \lor r) \land (\neg p \lor q)$
Consensus Laws	$(p \land q) \lor (q \land r) \lor (\neg p \land r) \equiv$ $(p \land q) \lor (\neg p \land r)$	$(p \lor q) \land (q \lor r) \land (\neg p \lor r) \equiv$ $(p \lor q) \land (\neg p \lor r)$
Tautology Laws ($\top$)	$p \lor (\top) \equiv \top$ $p \lor \neg p \equiv \top$ (True)	$p \land (\top) \equiv p$
	$\neg(\top) = \bot$	
Contradiction Laws ($\bot$)	$p \lor (\bot) \equiv p$	$p \land (\bot) \equiv \bot$ $p \land \neg p \equiv \bot$ (False)
	$\neg(\bot) \equiv \top$	
Exclusive Or Laws ($\oplus$)	$p \oplus q \equiv (p \lor q) \lor \neg(p \land q)$	$p \oplus q \equiv (\neg p \land q) \lor (p \lor \neg q)$

The Sixteen Logical Operations on Two Variables

#	Venn	Sym	Logical Notation(s)	Name(s)
0000		$\perp$	0	Contradiction; falsehood; antilogy; constant 0
0001		$\wedge$	$x \wedge y, \quad xy, \quad x \,\&\, y$	Conjunction; AND
0010		$\bar{\supset}$	$x \wedge \bar{y}, x \not\supset y, [x > y], x \dot{-} y$	Nonimplication; difference; but not
0011		L	x	Left projection
0100		$\bar{\subset}$	$\bar{x} \wedge y, x \not\subset y, [x < y], y \dot{-} x$	Converse nonimplication; not ... but
0101		R	y	Right projection
0110		$\oplus$	$x \oplus y, \quad x \not\equiv y, \quad x\char`^y$	Exclusive disjunction; nonequivalence; XOR
0111		$\vee$	$x \vee y, \quad x \mid y$	(Inclusive) disjunction; and/or; OR
1000		$\bar{\vee}$	$\bar{x} \wedge \bar{y}, \overline{x \vee y}, x \,\bar{\vee}\, y, x \downarrow y$	Nondisjunction; joint denial; neither... NOR
1001		$\equiv$	$x \equiv y, \quad x \leftrightarrow y, \quad x \Leftrightarrow y$	Equivalence; if and only if; IFF
1010		$\bar{R}$	$\bar{y}, \quad \neg y, \quad !\,y, \quad \sim y$	Right complementation; NOT
1011		$\subset$	$x \vee \bar{y}, x \subset y, x \Leftarrow y, [x \geq y], x^y$	Converse implication; IF
1100		$\bar{L}$	$\bar{x}, \quad \neg x, \quad !\,x, \quad \sim x$	Left complementation; NOT
1101		$\supset$	$\bar{x} \vee y, x \supset y, x \Rightarrow y, [x \leq y], y^x$	Implication; only if; if … then
1110		$\bar{\wedge}$	$\bar{x} \vee \bar{y}, \overline{x \wedge y}, x \,\bar{\wedge}\, y, x \mid y$	Nonconjunction; not both … and; NAND
1111		T	1	Affirmation; validity; tautology; constant 1

Donald E. Knuth (1968). 7.1.1 Boolean Basics, *The Art of Computer Programming*, Pre-fascicle 0B: The sixteen logical operations in two variables. See also Wikipedia, Truth function.

Logical Conditional Connective Laws

Law or Statement	Logical Expression	Is Equivalent To (≡)	Description
Conditional Laws	$p \rightarrow q$	$\neg p \lor q$ $\neg(p \land \neg q)$ Logical Equivalences: $p \lor q \equiv \neg p \rightarrow q$ $p \land q \equiv \neg(p \rightarrow \neg q)$ $\neg(p \rightarrow q) \equiv p \land \neg q$ $(p \rightarrow q) \land (p \rightarrow r) \equiv p \rightarrow (q \land r)$ $(p \rightarrow q) \lor (p \rightarrow r) \equiv p \rightarrow (q \lor r)$ $(p \rightarrow r) \land (q \rightarrow r) \equiv (p \land q) \rightarrow r$ $(p \rightarrow r) \lor (q \rightarrow r) \equiv (p \lor q) \rightarrow r$	Conditional, If ... Then, Implication
Biconditional Laws	$p \leftrightarrow q$	$(p \rightarrow q) \land (q \rightarrow p)$ $(p \rightarrow q) \land (\neg p \rightarrow \neg q)$ $(p \land q) \lor (\neg p \land \neg q)$ $\neg p \leftrightarrow \neg q$ Logical Equivalences: $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	Bi-conditional, If and only If, iff, XNOR
Sufficient Condition	p is a sufficient condition for q	The truth of p suffices to guarantee the truth of q.	
Necessary Condition	q is a necessary condition for p	For p to be true, it is necessary for q to be true also. $\neg q \rightarrow \neg p$	
Equivalence	$p \leftrightarrow q$	$p \equiv q$ $p \Longrightarrow q$	Is logically equivalent to ($p \equiv \neg$ $\neg p$) Is equivalent to
Contrapositive	$p \rightarrow q$	$\equiv \neg q \rightarrow \neg p$	True
Converse*	$p \rightarrow q$	$\not\equiv q \rightarrow p$	False
Inverse*	$p \rightarrow q$	$\not\equiv \neg p \rightarrow \neg q$	False

Rules of Inference with Propositions

Rule Name	Rule Logic	Example
Hypothesis	Givens. First lines of a proof.	It is raining today. You live in McKinney, Texas.
Therefore	$\therefore$	Therefore. In conclusion.
Modus Ponens	p $\underline{p \rightarrow q}$ $\therefore q$	It is raining today. If it is raining today, I will not ride my bike to school. Therefore, I will not ride my bike to school.
Modus Tollens	$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	If Sam studied for his test, then Sam passed his test. Sam did not pass his test. Therefore, Sam did not study for his test.
Addition, Generalization	$\underline{p}$ $\therefore p \vee q$	It is raining today. Therefore, it is either It is raining today or snowing today or both.
Simplification, Specialization	$\underline{p \wedge q}$ $\therefore p$	It is rainy today and it is windy today. Therefore, it is rainy today.
Conjunction	p $\underline{q}$ $\therefore p \wedge q$	Sam studied for his test. Sam passed his test. Therefore, Sam studied for his test and Sam passed his test.
Hypothetical Syllogism, Transitivity	$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	If you are mad then you will yell. If you yell then you will wake the baby. Therefore, if you are mad then you will wake the baby.
Disjunctive Syllogism, Elimination	$p \vee q$ $\underline{\neg p}$ $\therefore q$	Sam studied for his test or Sam took a nap. Sam did not study for his test. Therefore, Sam took a nap.
Resolution	$p \vee q$ $\underline{\neg p \vee q}$ $\therefore q \vee r$	Your shirt is red or your pants are blue. Your shirt is not red or your pants are blue. Therefore, your pants are blue or your shoes are white.
Proof by Division into Cases	$p \vee q$ $p \rightarrow r$ $\underline{q \rightarrow r}$ $\therefore r$	It is raining or it is Monday. It is raining so it is wet. It is Monday so it is wet. It is wet.
Contradiction Rule	$\underline{\neg p \rightarrow F}$ $\therefore p$	If it is not raining is a false statement, then it is raining.

Logical Predicates

Definition	Logical Expression	Is Equivalent To (≡)	Description
Universe of Discourse	***U***	All possible inputs in a given range	• Universe of Discourse • Universal Set • Universe
Domain of Discourse	$\mathbb{D}$	All possible inputs in a given range	• Domain of Discourse • Universe of Discourse
Proposition or Logical Statement	*p: "Roxy is a mammal"*	*p*	• Must be True or False • Cannot be a question • Cannot be a command
Predicate	*P(x): "x is a mammal"*	*P(x)*	• A logical statement whose truth value is a function of one or more <u>variables</u> • Truth depends upon the input variable *x* • P(x) ≠ a number • P(5) is a proposition
Example Statements	*q*: $\forall x \in \mathbb{D}$, *P(x)*: "x is a mammal"	*"For all x in the domain of discourse, P(x) is a mammal."*	• Is either True or False • A quantified predicate turns it into a logical statement
	T(x, y)	"*x* is a twin of *y*."	Predicate with two input variables
Truth Set (Single Free Variable)	*T = P(x)*	$T = \{a \mid P(a)\}$ $T = \{a \in A \mid P(a)\}$ $a \in T$	The set of all values of x that make the statement *p(x)* true
	Example:	$P(x_1)$, $P(x_2)$, and $P(x_3)$ are True	
Truth Set (Ordered Pair)	*T = P(x, y)*	$\{(a, b) \in A \times B \mid P(a, b)\}$ $(a, b) \in T$	Cross product truth set
	Examples:	$\{(p, n) \in P \times \mathbb{N} \mid$ the person p has n children$\} = \{$(John, 2), …$\}$ $\{(p, c, n) \in P \times C \times \mathbb{N} \mid$ the person p has lived in the city c for n years$\}$	

Logical Quantifiers

<table>
<tr><th>Definition</th><th>Logical Expression</th><th>Is Equivalent To (≡)</th><th>English</th></tr>
<tr><td>Universal Quantifier (∀)</td><td>$\forall x\ P(x)$
$\forall x \in P(x)$
$\forall x \in \mathbb{D},\ P(x)$

∀x, if x is in $\mathbb{D}$ then P(x)</td><td>“For all x in the domain, P(x) is true”

$\forall x \in A\ P(x) \equiv \forall x\ (x \in A \rightarrow P(x))$

For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$,
$\forall x\ P(x) \equiv P(a_1) \land P(a_2) \land \ldots \land P(a_k)$</td><td>for all
all elements
for each member
any
every
everyone
everybody
everything
x could be anything at all</td></tr>
<tr><td>Existential Quantifier (∃)</td><td>$\exists x\ P(x)$
$\exists x \in P(x)$
$\exists x \in \mathbb{D},\ P(x)$</td><td>“There exists x in the domain, such that P(x) is true”

For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$,
$\exists x\ P(x) \equiv P(a_1) \lor P(a_2) \lor \ldots \lor P(a_k)$

$P(x) \neq \emptyset$</td><td>there exists an x
there is
some
someone
somebody
at least one value of x
there is at least one x
it is the case that
the truth set is not equal to ∅</td></tr>
<tr><td>Uniqueness Quantifier (∃!)</td><td>$\exists! x\ P(x)$</td><td>there is a unique x in P(x) such that …

$\exists x\ (P(x) \land \neg\ y\ (P(y) \land y \neq x))$
$\exists x\ (P(x) \land \forall y\ (P(y) \rightarrow y = x))$
$\exists x\ \forall y\ (P(y) \leftrightarrow y = x)$

$\exists x\ P(x) \land \forall y\ \forall z((P(y) \land P(z)) \rightarrow y = z)$</td><td>unique
there is a unique x
there exists exactly one
there is exactly one x such that P(x)</td></tr>
<tr><td rowspan="2">Negated Existential Quantifier</td><td>$\neg\ [\exists x\ P(x)]$</td><td>$\forall x\ \neg P(x)$</td><td rowspan="2">nobody
no one
not one
there does not exist</td></tr>
<tr><td>$\neg\ [\forall x\ P(x)]$</td><td>$\exists x\ \neg P(x)$</td></tr>
<tr><td>Order of Precedence</td><td colspan="2">PEMDAS for Logic:
Parenthesis ()
Logical NOT (¬)
Quantifiers (∀, ∃)
Logical AND (∧)
Logical OR (∨)
Logical Conditional (→)
Logical Biconditional (↔)</td><td>Applied Left to Right

Example :
$\forall x\ P(x) \land Q(x) \equiv$
$(\forall x\ P(x)) \land Q(x)$</td></tr>
</table>

Quantifier Laws

Definition	Logical Expression	Is Equivalent To (≡)	Description / Example / • English
Abbreviation	$\exists x\,(x \in A \land \neg P(x))$	$\exists x \in A\ \neg P(x)$	Simplification
Expanding Abbreviation	$\forall x \in A\ P(x)$	$\forall x\,(x \in A \to P(x))$	Complication
Quantifier Negation Laws	$\forall x\ \neg P(x)$	$\neg \exists x\ P(x)$	• nobody's perfect
	$\neg \forall x\ P(x)$	$\exists x\ \neg P(x)$	• not everyone is perfect • someone is imperfect
Conditional Law	$x \in A \to P(x)$	$x \notin A \lor P(x)$	$p \to q \equiv \neg p \lor q$
Subset Negation Law	$x \in A$	$\neg(x \notin A)$	Swap ∈ with ∉, or vice versa
De Morgan's Law (Quantifier Negation)	$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$ $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$ $\neg\forall x\ \forall y\ P(x, y) \equiv \exists x\ \exists y\ \neg P(x, y)$ $\neg\forall x\ \exists y\ P(x, y) \equiv \exists x\ \forall y\ \neg P(x, y)$ $\neg\exists x\ \forall y\ P(x, y) \equiv \forall x\ \exists y\ \neg P(x, y)$ $\neg\exists x\ \exists y\ P(x, y) \equiv \forall x\ \forall y\ \neg P(x, y)$		De Morgan's Law for single and nested quantifiers
Nested / Multiple-Quantified Statements	$\forall x\ \forall y$	$\forall y\ \forall x$	• for all objects x and y, …
	$\exists x\ \exists y$	$\exists y\ \exists x$	• there are objects x and y such that …
	$\forall x\ \exists y\ P(x, y) \not\equiv \exists x\ \forall y\ P(x, y)$		False Counterexample for $x, y \in \mathbb{Z}$: $\forall x\ \exists y\ (x + y = 0) \equiv$ True $\exists x\ \forall y\ (x + y = 0) \equiv$ False
	$\neg(\forall x\ \exists y\ P(x, y))$	$\exists x\ \forall y\ \neg P(x, y)$	Negation of multiply-quantified statements
	$\neg(\exists x\ \forall y\ P(x, y))$	$\forall x\ \exists y\ \neg P(x, y)$	
Moving Quantifiers	$\forall x\ (P(x) \to \exists y\ Q(x, y)) \equiv$ $\forall x\ \exists y\ (P(x) \to Q(x, y))$		You can move a quantifier left if the variable is not used yet

Quantifier Logic Examples

Action	Logical Statement	English
Everyone	$\forall x\ \forall y\ P(x, y)$ NOTE: includes (x = y)	• everyone <did something> to everyone
Everyone Else	$\forall x\ \forall y\ (x \neq y) \to P(x, y)$ NOTE: excludes (x = y)	• everyone <did something> to everyone else
Someone Else	$\forall x\ \exists y\ ((x \neq y) \land P(x, y))$ NOTE: excludes (x = y)	• everyone <did something> to someone else
Exactly One	$\exists x\ (P(x) \land \forall y\ ((x \neq y) \to \neg P(y))) \equiv$ $\exists! x\ P(x)$	• exactly one person <did something>
No One	$\neg\exists x\ (P(x)$	• no one <did something>

Valid Quantifier Formulas

A		B
∀x (*P(x)* ∧ *Q(x)*)	≡	(∀x *P(x)* ∧ ∀x *Q(x)*)
∃x (*P(x)* ∧ *Q(x)*)	→	(∃x *P(x)* ∧ ∃x *Q(x)*)
∀x (*P(x)* ∨ *Q(x)*)	←	(∀x *P(x)* ∨ ∀x *Q(x)*)
∃x (*P(x)* ∨ *Q(x)*)	≡	(∃x *P(x)* ∨ ∃x *Q(x)*)
∀x (*P(x)* → *Q(x)*)	←	(∃x *P(x)* → ∀x *Q(x)*)
∃x (*P(x)* → *Q(x)*)	≡	(∀x *P(x)* → ∃x *Q(x)*)
∀x ¬*P(x)*	≡	¬∃x *P(x)*
∃x ¬*P(x)*	≡	¬∀x *P(x)*
∀x ∃y *T(x, y)*	←	∃y ∀x *T(x, y)*
∀x ∀y *T(x, y)*	≡	∀y ∀x *T(x, y)*
∃x ∃y *T(x, y)*	≡	∃y ∃x *T(x, y)*
∀x (*P(x)* ∨ *R*)	≡	(∀x *P(x)* ∨ *R*)
∃x (*P(x)* ∧ *R*)	≡	(∃x *P(x)* ∧ *R*)
∀x (*P(x)* → *R*)	≡	(∃x *P(x)* → *R*)
∃x (*P(x)* → *R*)	→	(∀x *P(x)* → *R*)
∀x (*R* → *Q(x)*)	≡	(R → ∀x *Q(x)*)
∃x (*R* → *Q(x)*)	→	(R → ∃x *Q(x)*)
∀x *R*	←	*R*
∃x *R*	→	*R*

Note: The above formulas are valid in classical first-order logic assuming that *x* does not occur free in *R*.

Invalid Quantifier Formulas

A		B	Counterexample
∃x (*P(x)* ∧ *Q(x)*)	←	(∃x *P(x)* ∧ ∃x *Q(x)*)	D = {*a*, *b*}, M = {*P(a)*, *Q(b)*}
∀x (*P(x)* ∨ *Q(x)*)	→	(∀x *P(x)* ∨ ∀x *Q(x)*)	D = {*a*, *b*}, M = {*P(a)*, *Q(b)*}
∀x (*P(x)* → *Q(x)*)	→	(∃x *P(x)* → ∀x *Q(x)*)	D = {*a*, *b*}, M = {*P(a)*, *Q(a)*}
∀x ∃y *T(x, y)*	→	∃y ∀x *T(x, y)*	D = {*a*, *b*}, M = {*T(a, b)*, *T(b, a)*}
∃x (*P(x)* → *R*)	←	(∀x *P(x)* → *R*)	D = Ø, M = {*R*}
∃x (*R* → *Q(x)*)	←	(*R* → ∃x *Q(x)*)	D = Ø, M = Ø
∀x *R*	→	*R*	D = Ø, M = Ø
∃x *R*	←	*R*	D = Ø, M = {*R*}

Note: if empty domains are not allowed, then the last four implications above are in fact valid.

Logical Truth Tables

p	q	Conjunction (and) ∧	NAND ⊼	Disjunction (or) ∨	NOR ⊽	XOR ⊻, ⊕	XNOR ⊙	Negation (not) ¬P
F	F	F	T	F	T	F	T	
F	T	F	T	T	F	T	F	T
T	F	F	T	T	F	T	F	F
T	T	T	F	T	F	F	T	

p	q	Material Implication (If … Then) →	Biconditional (Iff) ↔	Tautology (True) ⊤	Contradiction (False) ⊥
F	F	T	T	T	F
F	T	T	F	T	F
T	F	F	F	T	F
T	T	T	T	T	F

Blank Truth Tables

Inputs				Output		
p	q	r	s	x	y	z
F	F	F	F			
F	F	F	T			
F	F	T	F			
F	F	T	T			
F	T	F	F			
F	T	F	T			
F	T	T	F			
F	T	T	T			
T	F	F	F			
T	F	F	T			
T	F	T	F			
T	F	T	T			
T	T	F	F			
T	T	F	T			
T	T	T	F			
T	T	T	T			

Inputs			Output	
p	q	r	x	y
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

Inputs		Output
p	q	x
F	F	
F	T	
T	F	
T	T	

Sources

SNHU MAT 230 - Discrete Mathematics, zyBooks.

See also

- Harold's Sets Cheat Sheet
- Harold's Boolean Algebra Cheat Sheet
- Harold's Proofs Cheat Sheet

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Donald E. Knuth (1968). 7.1.1 Boolean Basics, *The Art of Computer Programming*, Pre-fascicle 0B: The sixteen logical operations in two variables.