Harold's Calculus
Cheat Sheet
6 November 2026

Limits & Continuity

Definition of Limit

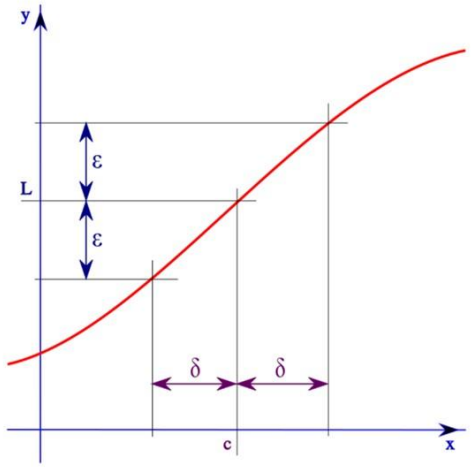
Let f be a function defined on an open interval near c, possibly excluding c.
Let L be a real number.

The statement:

$$\lim_{x\to c} f(x) = L$$

means that for each $\epsilon > 0$
there exists a $\delta > 0$ such that

if $0 < |x - c| < \delta$,
then $|f(x) - L| < \epsilon$.



The Existence of a Limit

The limit of $f(x)$ as x approaches c is L if and only if ($\Leftrightarrow$) the limit is the same L from both directions.

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

(Bonus if $\lim_{x\to c} f(x) = f(c) = L$, but not required.)

Continuous Functions

If $f(x)$ is continuous at c then $\lim_{x\to c} f(x) = f(c)$.

Definition of Continuity

A function $\boldsymbol{f}$ **is continuous** at c in the domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$ for every x in the domain.

Tip:
Rearrange $|f(x) - f(c)|$ to have $|(x - c)|$ as a factor. Since $0 < |x - c| < \delta$ we can find an equation that relates both δ and ε together.

Prove that $\boldsymbol{f(x) = x^2 - 1}$ is a continuous function.

Given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \varepsilon.$$

1. Simplify the difference

$$\begin{aligned} &|f(x) - f(c)| \\ &= |(x^2 - 1) - (c^2 - 1)| = |x^2 - 1 - c^2 + 1| \\ &= |x^2 - c^2| = |(x - c) \cdot (x + c)| \\ &= |x - c| \cdot |x + c| \end{aligned}$$

2. Bound the extra factor $|x + c|$

Require $|x - c| < 1$. Then

$$|x + c| = |(x - c) + 2c| \le |x - c| + 2|c| < 1 + 2|c|.$$

3. Choose

$$\delta = \min\left\{1, \frac{\varepsilon}{1 + 2|c|}\right\}.$$

4. Verify

If $|x - c| < \delta$, then

$$\begin{aligned} |f(x) - f(c)| &= |x - c|\,|x + c| \\ &\le |x - c|\,(1 + 2|c|) \\ &\boxed{< \delta(1 + 2|c|) \le \varepsilon} \end{aligned}$$

Therefore, f is continuous at every real c, including $c = 0$.
$f(x) = x^2 - 1$ is continuous for all real x.

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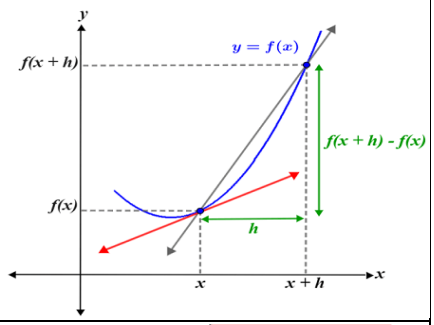
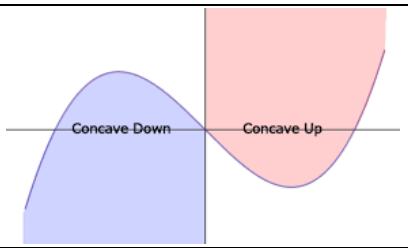
Limit Properties[1]	
1. Constant	$\lim_{x\to a} c = c$
2. Identity Property	$\lim_{x\to a} x = a$
3. Constant Multiple	$\lim_{x\to a}[c\,f(x)] = c\lim_{x\to a} f(x)$
4. Sum and Difference	$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
5. Product	$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
6. Quotient	$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
7. Exponent – Constant	$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
8. Exponent – Variable	$\lim_{x\to a}[f(x)]^{g(x)} = \left[\lim_{x\to a} f(x)\right]^{\lim_{x\to a} g(x)}$ $\lim f(x) = A > 0$ and $\lim g(x) = B \in \mathbb{R}$
9. n[th] Root	$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ For even n, $f(x) \geq 0$ near a and $\lim_{x\to a} f(x) \geq 0$.
10. Composite Function	$\lim_{x\to a}(f\circ g)(x) = \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$ f must be continuous at $\lim_{x\to a} g(x)$.

[1] Assume the individual limits exist and are finite, and all expressions are defined near a.

Derivative Notation		
First Derivative of a Function[1] (Slope Function) [1] This symmetric difference quotient only works under suitable conditions.	$\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$ $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$ $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$	
Second Derivative of a Function[1] (Concavity Function)	$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$ $= \lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$	

Notation	Leibniz	Lagrange	Newton	Euler
1st Derivative Notation	$f'(x) = \frac{d}{dx}f(x)$	y'	$\dot{x} = \frac{dx}{dt}$	$D_x[y]$
2nd Derivative Notation	$f''(x) = \frac{d^2}{dx^2}f(x)$	y''	$\ddot{x} = \frac{d^2x}{dt^2}$	$D_x^2 f(x)$
n[th] Derivative Notation	$f^{(n)}(x) = \frac{d^n}{dx^n}f(x)$	$y^{(n)}$	$x^{(n)} = \frac{d^n x}{dt^n}$	$D_x^n f(x)$

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Common Derivatives	(See Cengage Learning 1-Page Calculus Formulas)
1. Constant Rule	$\frac{d}{dx}[c] = 0$
2. Constant Multiple Rule	$\frac{d}{dx}[cf(x)] = cf'(x)$
3. Sum and Difference Rule	$\frac{d}{dx}[f \pm g] = f' \pm g'$
4. Product Rule	$\frac{d}{dx}[fg] = fg' + g\,f'$
5. Quotient Rule	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$
6. Chain Rule	$\frac{d}{dx}[(f \circ g)(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
7. Power Rule $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$\boldsymbol{\frac{d}{dx}[x^n] = nx^{n-1}}$ | $\frac{d}{dx}[cx^n] = ncx^{n-1}$ Can be used for roots if n is a fraction.
8. Power Rule for x	$\frac{d}{dx}[x] = 1$ (Think $x = x^1$ and $x^0 = 1$)
9. Power Rule with Chain Rule	$\frac{d}{dx}[f(x)^n] = nf(x)^{n-1}\,f'(x)$
10. Power Rule for Roots	$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}, \quad x > 0$
11. Power Rule for Roots with Chain Rule	$\frac{d}{dx}\left[\sqrt[n]{f(x)}\right] = \frac{1}{n\left(\sqrt[n]{f(x)}\right)^{n-1}} f'(x)$
12. Absolute Value	$\frac{d}{dx}\lvert x\rvert = \frac{x}{\lvert x\rvert}, \quad x \neq 0$
13. Natural Exponent	$\boldsymbol{\frac{d}{dx}[e^x] = e^x}$
14. Natural Exponent with Chain Rule	$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}f'(x)$
15. Exponential Rule	$\frac{d}{dx}[a^x] = (\ln a)\,a^x, \quad a > 0$
16. Exponential Rule with Chain Rule	$\frac{d}{dx}\left[a^{f(x)}\right] = (\ln a)\,a^{f(x)}\,f'(x), \quad a > 0$
17. Natural Logarithm	$\boldsymbol{\frac{d}{dx}[\ln x] = \frac{1}{x}}, \quad x > 0$
18. Natural Logarithm with Chain Rule	$\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} \cdot f'(x), \quad f(x) > 0$
19. Logarithm	$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln(a)}, \quad x > 0, a > 0, a \neq 1$ $\frac{d}{dx}[\log_a \lvert x\rvert] = \frac{1}{x\ln(a)}, \quad x \neq 0, a > 0, a \neq 1$
20. Logarithm with Chain Rule	$\frac{d}{dx}[\log_a f(x)] = \frac{1}{\ln a} \cdot \frac{f'(x)}{f(x)}, \quad a > 0,\ a \neq 1,\ f(x) > 0$

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21. Sine	$\boldsymbol{\frac{d}{dx}[\sin(x)] = \cos(x)}$, x is in radians for all trig functions
22. Cosine	$\boldsymbol{\frac{d}{dx}[\cos(x)] = -\sin(x)}$
23. Tangent	$\frac{d}{dx}[\tan(x)] = \sec^2(x)$
24. Cotangent	$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$
25. Secant	$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
26. Cosecant	$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$
27. Arcsine	$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$
28. Arccosine	$\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$
29. Arctangent	$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$
30. Arccotangent	$\frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1+x^2}$
31. Arcsecant	$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \quad \lvert x\rvert > 1$
32. Arccosecant	$\frac{d}{dx}[\csc^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}, \quad \lvert x\rvert > 1$
33. Hyperbolic Sine $\left(\frac{e^x-e^{-x}}{2}\right)$	$\frac{d}{dx}[\sinh(x)] = \cosh(x)$
34. Hyperbolic Cosine $\left(\frac{e^x+e^{-x}}{2}\right)$	$\frac{d}{dx}[\cosh(x)] = \sinh(x)$
35. Hyperbolic Tangent	$\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2(x)$
36. Hyperbolic Cotangent	$\frac{d}{dx}[\coth(x)] = -\operatorname{csch}^2(x)$
37. Hyperbolic Secant	$\frac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$
38. Hyperbolic Cosecant	$\frac{d}{dx}[\operatorname{csch}(x)] = -\operatorname{csch}(x)\coth(x)$
39. Hyperbolic Arcsine	$\frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$
40. Hyperbolic Arccosine	$\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$
41. Hyperbolic Arctangent	$\frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}, \quad \lvert x\rvert < 1$
42. Hyperbolic Arccotangent	$\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}, \quad \lvert x\rvert > 1$
43. Hyperbolic Arcsecant	$\frac{d}{dx}[\operatorname{sech}^{-1}(x)] = -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

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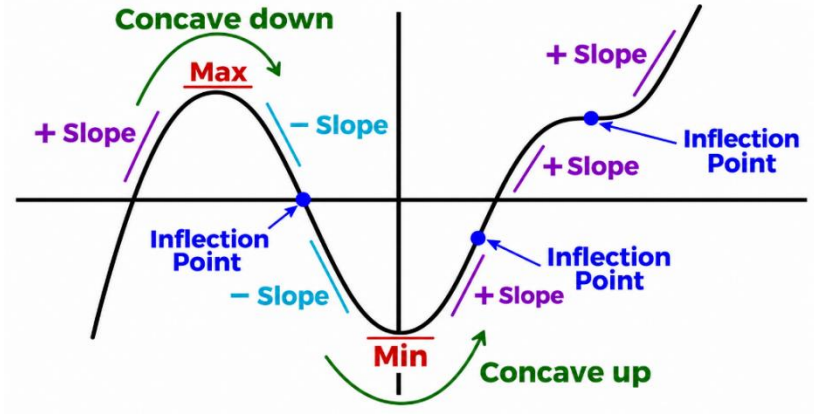
44. Hyperbolic Arccosecant	$\frac{d}{dx}[\operatorname{csch}^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}}, \qquad x \neq 0$

[1] The hyperbolic-function derivative material 33-44 provides optional enrichment.

Graphing with Derivatives				
f	**Gives Us**	**When set $= 0$**	**Graph Info**	**Physics**
$f(x)$	Height, y	Roots	Sketch known functions	Position, $s(t)$
$f'(x)$	Slope, m	Critical Numbers; Possible Local Extrema	Choose one test point in each interval separated by critical numbers or domain breaks. This determines where the function is increasing/decreasing. − ↘ ↗ +	Velocity, $v(t)$
$f''(x)$	Concavity	Candidates for Inflection Points	Concave up ∪ if + Concave down ∩ if −	Acceleration, $a(t)$

Analyzing the Graph of a Function	(See Harold's Graphing Rationals Cheat Sheet)
x-Intercepts (Zeros or Roots)	$f(x) = 0$
y-Intercept	$f(0) = y$
Domain	Valid x values
Range	Valid y values
Continuity	At c: $f(c)$ exists, $\lim_{x\to c} f(x)$ exists, and they are equal.
Vertical Asymptotes (VA)	$x = c$ is a vertical asymptote if at least one one-sided limit at c equals $+\infty$ or $-\infty$.
Horizontal Asymptotes (HA)	$\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to+\infty} f(x) = L$
Infinite Limits at Infinity	As $x \to \infty$ or $x \to -\infty$, $f(x)$ may approach $+\infty$ or $-\infty$. Check each end separately.
Differentiability	The left-hand and right-hand derivatives exist, are finite, and are equal.
Relative Extrema	Create a table with domains: $f(x), f'(x), f''(x)$
Concavity	If $f''(x) > 0$, then **cup up** ∪ If $f''(x) < 0$, then **cup down** ∩
Inflection Points	Candidates: $f''(c) = 0$ or $f''(c)$ is undefined, concavity must change
Graphing with Derivatives	

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Derivative Tests	
Test for Increasing and Decreasing Functions	Throughout an interval, 1. If $f'(x) \geq 0$, then f is **monotone increasing** (slope up) ↗ 2. If $f'(x) > 0$, then f is **strictly increasing** (slope up) ↗ 3. If $f'(x) = 0$, then f is **constant** throughout an interval (zero slope) → 4. If $f'(x) \leq 0$, then f is **monotone decreasing** (slope down) ↘ 5. If $f'(x) < 0$, then f is **strictly decreasing** (slope down) ↘
First Derivative Test	Critical points: Assume f is continuous at the critical number c. 1. If $f'(x)$ changes from – to + at c, then f has a *relative* **minimum** at $(c, f(c))$ 2. If $f'(x)$ changes from + to – at c, then f has a *relative* **maximum** at $(c, f(c))$ 3. If $f'(x)$ is positive on both sides or negative on both sides of c, then $f(c)$ is not a relative extremum Absolute minimum/maximum: 1. Test $f(x)$ at the domain boundaries $[a, b]$, meaning check $f(a)$ and $f(b)$ 2. Include $f(x)$ for all critical points as well 3. The largest/smallest wins
Second Derivative Test	Let $f'(c) = 0$, and $f"(c)$ exists, then 1. If $f''(c) > 0$, then f has a relative **minimum** at $(c, f(c))$ 2. If $f''(c) < 0$, then f has a relative **maximum** at $(c, f(c))$ 3. If $f''(c) = 0$, then the test is inconclusive
Test for Concavity	1. If $f''(x) > 0$ for all x, then the graph is concave up (**cup up** ∪) 2. If $f''(x) < 0$ for all x, then the graph is concave down (**cup down** ∩)
Inflection Points (Change in concavity)	An inflection point $(c, f(c))$ is a point on a continuous graph where concavity changes. Candidates occur where $f'' = 0$ or f'' is undefined.

Extrema	Definition
Local Maximum (Relative Max.)	A function $f(x)$ has a local max at $x = a$ if $f(a)$ is greater than or equal to (≥) the values of $f(x)$ in some interval around a.
Local Minimum (Relative Min.)	A function $f(x)$ has a local minimum at $x = a$ if $f(a)$ is less than or equal to (≤) the values of $f(x)$ in some interval around a.
Absolute Maximum (Global Max.)	A function $f(x)$ has an absolute maximum at $x = c$ if $f(c)$ is greater than or equal to (≥) $f(x)$ for all x in the domain of f.
Absolute Minimum (Global Min.)	A function $f(x)$ has an absolute minimum at $x = d$ if $f(d)$ is less than or equal to (≤) $f(x)$ for all x in the domain.
Absolute Extrema	Continuity on a closed interval guarantees an absolute maximum and minimum.
Critical Points	A critical number c belongs to the domain of f, and $f'(c) = 0$ or $f'(c)$ does not exist.
Endpoints	For absolute extrema on a closed interval $[a, b]$ evaluate the function at the critical points and at the endpoints $f(a)$ and $f(b)$. The largest value is the absolute maximum, and the smallest is the absolute minimum.

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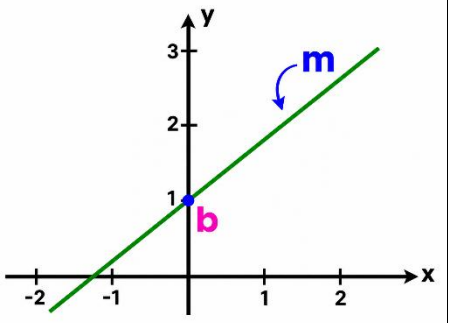
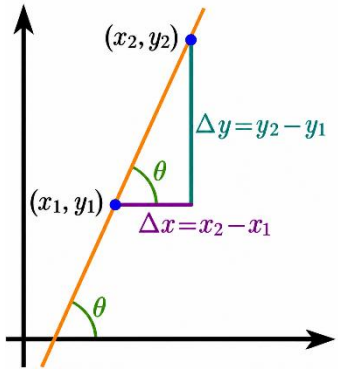
Line Form	Linear Equation	Graph
General Form	$ax + by + c = 0$ where a, b are not both zero	
Standard Form	$ax + by = c$	
Slope-Intercept Form	$y = mx + b$	
Point-Slope Form	$y - y_0 = m(x - x_0)$ For a tangent line to $y = f(x)$, use $m = f'(x_0)$ at point (x_0, y_0).	
Two-Point Form	$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$ $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ (no division by 0 and $x_2 \neq x_1$)	
Intercept Form	$\frac{x}{a} + \frac{y}{b} = 1$ where a is the non-zero $x-$ intercept and b is the non-zero $y-$ intercept	
Calculus Form (Tangent Line Form)	$L(x) = f'(c)(x - c) + f(c)$	
Slope	$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$	
Vertical Line	$x = a$	
Horizontal Line	$y = b$	

Physics	Translational Motion	
	Vector	**Scalar**
Position	$\boldsymbol{s}(t) = \boldsymbol{s}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2$ (constant acceleration)	$x(t) = x_0 + v_{0,x}t + \frac{1}{2}a_x t^2$ $y(t) = y_0 + v_{0,y}t + \frac{1}{2}gt^2$
Velocity	$\boldsymbol{v}(t) = \boldsymbol{s}'(t)$	$v(t) = v_0 + at$ $v^2 = v_0^2 + 2a(x - x_0)$ (constant acceleration)
Acceleration	$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{s}''(t)$	$a(t) = a$
Acceleration due to Gravity (Planet Earth)	$g \approx -9.81 \frac{m}{s^2}$	$g \approx -32.2 \frac{ft}{s^2}$
	Vertical acceleration due to gravity, upward positive.	

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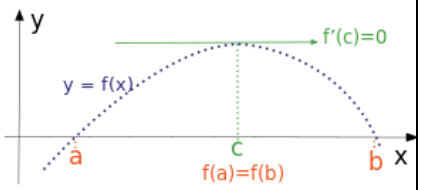
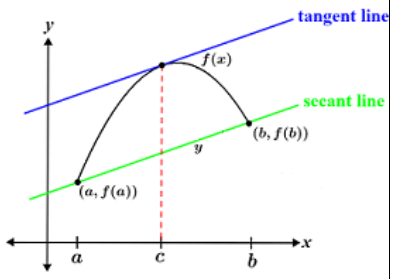
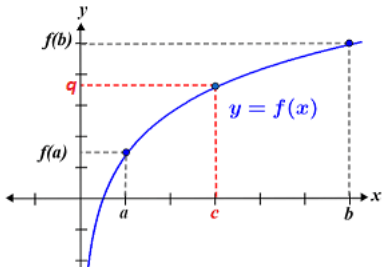
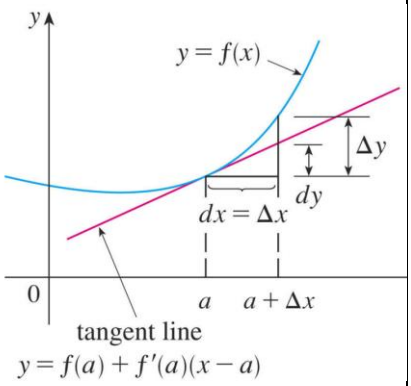
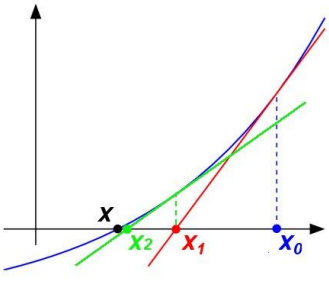
Differentiation and Differentials		
Rolle's Theorem	Assume f is <u>continuous</u> on the closed interval $[a, b]$, and f is <u>differentiable</u> on the open interval (a, b). If $f(a) = f(b)$, then there exists at least one number c in (a, b) such that $f'(c) = 0$.	
Mean Value Theorem (for derivatives)	Assume f is <u>continuous</u> on the closed interval $[a, b]$, and f is <u>differentiable</u> on the open interval (a, b), then at least one $c \in (a, b)$ exists. $f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{\Delta y}{\Delta x}$ $\boxed{f(b) = f(a) + (b - a)f'(c)}$	
Intermediate Value Theorem	Assume f is a <u>continuous</u> function with the interval $[a, b]$ as its domain. If f takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.	
Calculating Differentials (Tangent line approximation)	$dy = f'(x)\,dx$ $\Delta y \approx f'(x)\,\Delta x$ $f(x + \Delta x) = f(x) + \Delta y$ $f(x + \Delta x) \approx L(x + \Delta x)$ $\boxed{L(x + \Delta x) = f(x) + f'(x)\,\Delta x}$ Percent Relative Error: $\dfrac{\lvert L(x + \Delta x) - f(x + \Delta x)\rvert}{\lvert f(x + \Delta x)\rvert} \times 100\%$ $f(x + \Delta x) \neq 0$ Example: $\sqrt[4]{82}$ $f(x) = \sqrt[4]{x}, \quad x = 81, \quad \Delta x = 1$ $\sqrt[4]{82} \approx L(82) = f(81) + f'(81)(1)$ $= 3 + \dfrac{1}{108} \approx 3.00926$	
Newton Method (Newton-Raphson Method)	Finds zeros of f, or finds c if $f(c) = 0$. $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$ Convergence is not guaranteed. Example: $\sqrt[4]{82}$ $f(x) = x^4 - 82 = 0$ Start with $x_0 = 3$.	

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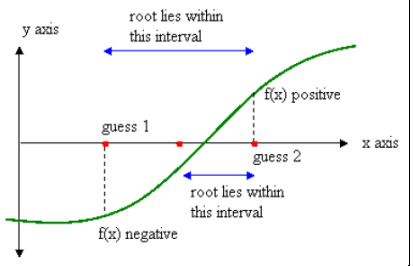
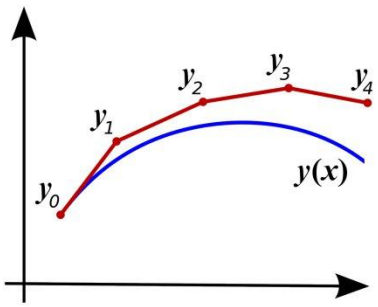
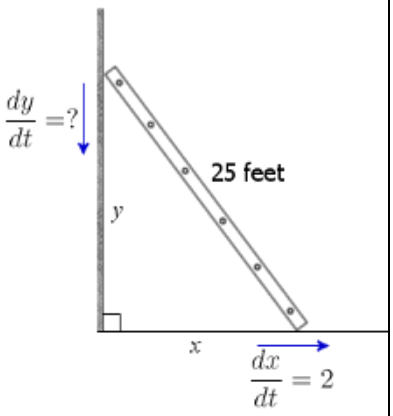
Bisection Method	Finds zeros of f for a continuous function. 1. Find two points, a and b, with opposite endpoint signs, where $f(a) \cdot f(b) < 0$. 2. Calculate the midpoint $t = (a + b)/2$. 3. If $f(t) = 0$, then t is the root of the function. $f(t) = 0$ may never occur. 4. Divide the interval $[a, b]$ in half and keep the half-interval whose endpoint function values have opposite signs. 5. Repeat the process until desired accuracy.	
Euler's Method	Approximates a function $y(x)$ given its derivative function y', a starting point, and small steps. Derivative function: $y' = f(x, y)$ Initial condition: $y_0 = y(x_0)$ Approx. values: $y_n \approx y(x_n)$ Step size: Δx $x_{n+1} = x_n + \Delta x$ $\boxed{y_{n+1} = y_n + \Delta x\, f(x_n, y_n)}$	
Related Rates	1. Identify the known variables and rates of change. $x = 15\ ft, y = 20\ ft, ladder\ r = 25\ ft$ $x' = 2\frac{ft}{s}, \quad y' = ____\frac{ft}{s}$ 2. Construct an equation relating these quantities. $x^2 + y^2 = r^2$ (Typically, the Pythagorean Theorem, similar triangles, or volume formulas.) 3. Differentiate both sides of the equation. $2xx' + 2yy' = 0$ 4. Solve for the desired rate of change. $y' = -\frac{x}{y}\, x'$ 5. Substitute the known rates of change and quantities into the equation. $y' = -\frac{15}{20} \bullet 2 = -1.5\ \frac{ft}{s}$	
L'Hôpital's Rule	If $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{P(x)}{Q(x)} \to \left\{\frac{0}{0}, \frac{\infty}{\infty}\right\}$, then $\boxed{\lim_{x\to c} \frac{P(x)}{Q(x)} = \lim_{x\to c} \frac{P'(x)}{Q'(x)} = \lim_{x\to c} \frac{P''(x)}{Q''(x)} = \cdots}$ Check before each application that $Q'(x) \neq 0$. Can transform from $\{0 \bullet \infty, \infty - \infty, 0^0, 1^\infty, \infty^0\}$ to $\{0/0, \infty/\infty\}$, but not from $\{0^\infty, \infty^\infty\}$.	

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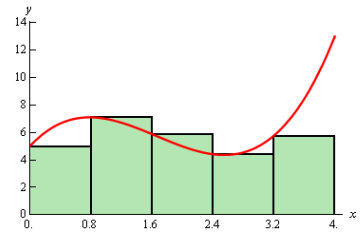
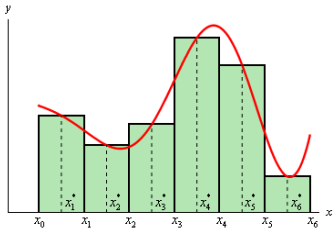
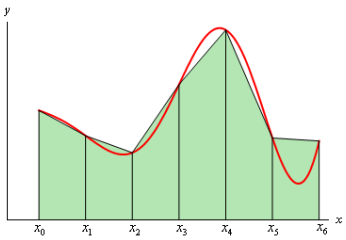
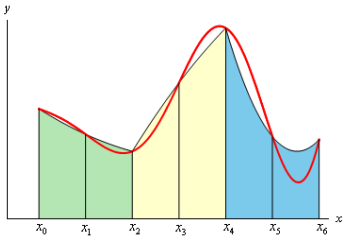
Numerical Methods	(See Harold's Numerical Analysis Algorithms in C)	
Riemann Sum	$$I = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ Types: • Left Sum (LHS) • Middle Sum (MHS) • Right Sum (RHS)	
Midpoint Rule (Middle Sum/MHS) (0th degree polynomial approx.)	$$I = \int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x =$$ $$\Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \cdots + f(\bar{x}_n)]$$ where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{(x_{i-1} + x_i)}{2}$ = midpoint of $[x_{i-1}, x_i]$ Error Bounds: $\lvert E_M\rvert \le \frac{K(b-a)^3}{24n^2}$, $K \ge \max_{x\in[a,b]} \lvert f''(x)\rvert$	
Trapezoidal Rule (1st degree polynomial approx.)	$$I = \int_a^b f(x)\,dx \approx T_n =$$ $$\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $\lvert E_T\rvert \le \frac{K(b-a)^3}{12n^2}$, $K \ge \max_{x\in[a,b]} \lvert f''(x)\rvert$	
Simpson's Rule (2nd degree polynomial approx.)	$$I = \int_a^b f(x)\,dx \approx S_n =$$ $$\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$ where n is even and $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $\lvert E_S\rvert \le \frac{K(b-a)^5}{180n^4}$, $K \ge \max_{x\in[a,b]} \lvert f^{(4)}\rvert$	
TI-84 Plus (Numeric)	**[MATH] [9:fnInt(** f(X), X, a, b) **[MATH]** [1] **[ENTER]** Example: **[MATH] [9:fnInt(**X^2, X, 0, 1)] **[ENTER]** Optionally **[MATH] [1:▶Frac] [ENTER]** → **1/3** $$\int_0^1 x^2\,dx = \frac{1}{3}$$	
TI-Nspire CAS (Algebraic)	**[MENU]** [4] Calculus [3] Integral **[TAB] [TAB]** [X] [^] [2] **[TAB]** **[TAB]** [X] **[ENTER]** Shortcut: **[ALPHA] [WINDOWS]** [4]	

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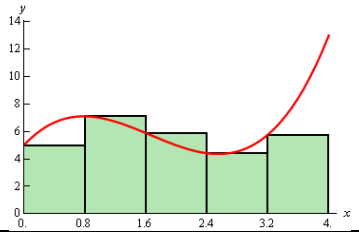
Integration Properties and Definitions	
Basic Integration Properties	Integral of a derivative $\int f'(x)\,dx = f(x) + C$
	Derivative of an integral $\frac{d}{dx}\int f(x)\,dx = f(x) = F'(x)$
	Integration is the "inverse" of differentiation, and vice versa. **Tip**: Use the $f'(x)$ tables and integrate both sides to determine many common integrals.
Riemann Sum	$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \qquad \text{where } x_{i-1} \le c_i \le x_i$ 
Definition of a Definite Integral (Area under the curve or signed/net area)	$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$ $\|\Delta\| = max_i\,\Delta x_i$ or $\|\Delta\| = \Delta x = \frac{b-a}{n}$ for equal subdivisions
Indefinite Integral	$\int f(x)\,dx = F(x) + C$ (F is an antiderivative of f)
Definite Integral	$\int_a^b f(x)\,dx = F(b) - F(a)$
Improper Integral	$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
Swap Bounds	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Additive Interval Property	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

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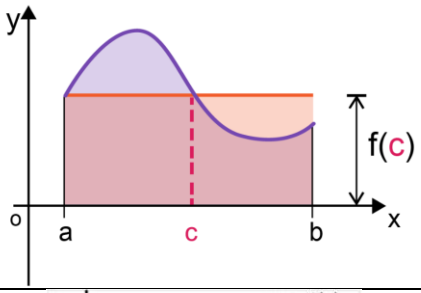
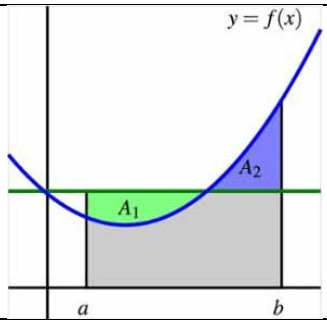
Integration Theorems	(See Harold's Fundamental Theorem of Calculus Cheat Sheet) (See Harold's Calculus BC Formulas Cheat Sheet)
First Fundamental Theorem of Calculus	$$F' = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$ $$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f\big(g(x)\big)g'(x)$$ $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$$ Assume f is continuous and the variable bounds g, h are differentiable.
Second Fundamental Theorem of Calculus	$$\int_a^b f(x)\,dx = F(b) - F(a)$$ Antiderivative evaluated at the upper bound minus the lower bound. Assumes continuity of $f(x)$.
Net Change Theorem (Displacement)	$$\int_a^b f'(t)\,dt = f(b) - f(a)$$ Example: $$\int_a^b v(t)\,dt = s(b) - s(a),\ \text{where } v(t) = s'(t) = \frac{d\,s(t)}{dt}$$
Total Variation Theorem (Distance Traveled)	$$\int_a^b |f'(t)|\,dt$$
Mean Value Theorem (for Integrals)	$$\int_a^b f(x)\,dx = f(c)(b-a)$$ There exists $c \in (a, b)$. (assumes continuity) 
Average Value of a Function	Continuous: $$\frac{1}{b-a}\int_a^b f(x)\,dx$$ Discrete: $$\text{Average} = \frac{1}{n}\sum_{i=1}^{n} a_i$$ 

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Common Antiderivatives	(See Cengage Learning 1-Page Calculus Formulas)
1. Zero Rule $f(x) = 0$	$\int 0\,dx = C$
2. Constant Rule $f(x) = k = kx^0$	$\int k\,dx = kx + C$ | $\int \boldsymbol{dx = x + C}$
3. Constant Multiple Rule	$\int k\,f(x)\,dx = k\int f(x)\,dx$
4. Sum and Difference Rule	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
5. Power Rule $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$\int \boldsymbol{x^n\,dx = \frac{x^{n+1}}{n+1} + C}$, where $n \neq -1$ *If* $n = -1$, then $\int x^{-1}\,dx = ln\|x\| + C$
6. Power Rule w/Chain Rule	If $u = g(x)$, and $u' = \frac{d}{dx}g(x)$ then $\int u^n\,\boldsymbol{u'}dx = \frac{u^{n+1}}{n+1} + C$, where $n \neq -1$
7. Natural Exponent	$\int \boldsymbol{e^x\,dx = e^x + C}$
8. Exponent	$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \quad a > 0,\ a \neq 1$
9. Natural Logarithm	$\int \boldsymbol{\frac{1}{x}\,dx = ln\|x\| + C}, \quad x \neq 0$
10. Logarithm	$\int \log_a\|x|\,dx = x\log_a\|x| - \frac{x}{\ln a} + C, \quad a > 0,\ a \neq 1,\ x \neq 0$
11. Sine	$\int \boldsymbol{sin(x)\,dx = -cos(x) + C}$
12. Cosine	$\int \boldsymbol{cos(x)\ dx = sin(x) + C}$
13. Tangent	$\int \tan(x)\,dx = -\,ln\,\|cos(x)\| + C$
14. Cotangent	$\int \cot(x)\,dx = ln\,\|sin(x)\| + C$
15. Secant	$\int \sec(x)\,dx = ln\,\|sec(x) + \tan(x)\| + C$
16. Cosecant	$\int \csc(x)\,dx = -\,ln\,\|csc(x) + cot(x)\| + C$
17. Secant^2	$\int sec^2(x)\ dx = tan(x) + C$
18. Cosecant^2	$\int csc^2(x)\,dx = -\,cot(x) + C$
19. Arcsine	$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$
20. Arctangent	$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\,tan^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$
21. Arcsecant	$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\,sec^{-1}\left(\frac{\|x\|}{a}\right) + C, \quad a > 0$

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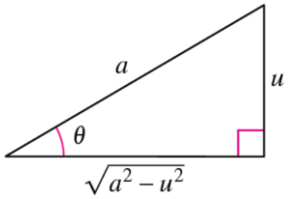
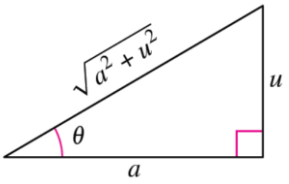
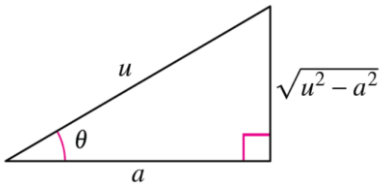
Integration Methods	
1. Memorized	See Cengage Learning 1-Page Calculus Formulas
2. U-Substitution	$\int f(g(x))g'(x)dx = F(g(x)) + C$ Set $u = g(x)$, then $du = g'(x)\,dx$ $\int f(u)\,du = F(u) + C$ $u =$ _____ $du =$ _____ dx
3. Integration by Parts	$\int u\,dv = uv - \int v\,du$ $u =$ _____ $v =$ _____ $du =$ _____ dx $dv =$ _____ dx The **LIATE** Rule often helps choose u. **L – Logarithmic**: $\ln x\,, log_b\,x$ **I – Inverse Trig.**: $sin^{-1}(x)\,, cos^{-1}(x)\,, tan^{-1}(x)$ **A – Algebraic**: $x^2, 3x^{60}, \sqrt{x}$ **T – Trigonometric**: $sin(x)\,, cos(x)\,, tan(x)$ **E – Exponential**: $e^x, 19^x$
4. Partial Fractions	$\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ and $Q(x)$ are polynomials and deg $P <$ deg Q.
5a. Trig Substitution for $\sqrt{a^2 - u^2}$	$\int \sqrt{a^2 - u^2}\,du$ $a > 0$ Substitution: $u = a\,sin\,\theta$ Identity: $1 - sin^2\,\theta = cos^2\,\theta$
5b. Trig Substitution for $\sqrt{u^2 + a^2}$	$\int \sqrt{u^2 + a^2}\,du$ $a > 0$ Substitution: $u = a\,tan\,\theta$ Identity: $tan^2\,\theta + 1 = sec^2\,\theta$
5c. Trig Substitution for $\sqrt{u^2 - a^2}$	$\int \sqrt{u^2 - a^2}\,du$ $a > 0$ Substitution: $u = a\,sec\,\theta$ Identity: $sec^2\,\theta - 1 = tan^2\,\theta$
6. Computer Algebra System (CAS)	TI-Nspire CX CAS Graphing Calculator TI –Nspire CAS iPad app
7. Numerical Methods	Riemann Sum, Midpoint Rule, Trapezoidal Rule, Simpson's Rule, various quadrature rules, TI-84 Calculator, etc.
8. WolframAlpha	WolframAlpha is the Google of mathematics. Shows steps.
9. AI Chatbot	OpenAI ChatGPT, Anthropic Claude, Microsoft Copilot, Google Gemini, SpaceX AI Grok, Perplexity.AI, others







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Partial Fractions	(See Harold's Partial Fraction Decomposition Cheat Sheet)
Condition	$$f(x) = \frac{P(x)}{Q(x)}$$ $P(x)$ and $Q(x)$ are polynomials with $\deg P < \deg Q$. If $\deg P \geq \deg Q$, do long division first.
Example Expansion	$$\frac{P(x)}{(ax+b)(cx+d)^2(ex^2+fx+g)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{(ex^2+fx+g)}$$ Assume $ax+b$ and $cx+d$ have distinct roots, $a, c, e \neq 0$, and ex^2+fx+g is irreducible over the reals.
Typical Solution	$$\int \frac{a}{x+b}\,dx = a\,ln\lvert x+b\rvert + C$$

Taylor Series	(See Harold's Taylor Series Cheat Sheet) (See Harold's Infinite Series Cheat Sheet)
Taylor Polynomial with Remainder	$$f(x) = P_n(x) + R_n(x)$$ $$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}}$$ $$\lvert R_n(x)\rvert \leq \frac{M\lvert x-c\rvert^{n+1}}{(n+1)!}$$ where $\left\lvert f^{(n+1)}(t)\right\rvert \leq M$ for every t between c and x. Assume f has continuous derivatives through order $n+1$ on an interval containing c and x. The number ξ lies between c and x.
Infinite Taylor Series	The Taylor series represents $f(x)$ wherever $R_n(x) \to 0$ as $n \to \infty$.

Sequences and Series	(See Harold's Series Cheat Sheet)
Sequence	$a_1, a_2, \ldots a_{n-1}, a_n, \ldots$
Convergent Sequence	$\lim_{n\to\infty} a_n = L$ (Limit) Example: $a_n = 1/n$, so $\lim_{n\to\infty} 1/n = 0$
Geometric Series	$$S_n = \frac{a(1-r^n)}{1-r}, \qquad r \neq 1$$ $$S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$$ and converges when $\lvert r\rvert < 1$

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Convergence Tests	(See Harold's Series Convergence Tests Cheat Sheet)	
Series Convergence Tests	1. Divergence or n^{th} Term 2. Geometric Series 3. p-Series 4. Alternating Series 5. Integral	6. Ratio 7. Root 8. Direct Comparison 9. Limit Comparison 10. Telescoping Series

Uncommon Definite Integrals	
22. Wallis Cosine Formula	$$\int_0^{\frac{\pi}{2}} cos^n x \ dx = \int_0^{\frac{\pi}{2}} sin^n x \ dx$$ $$= \frac{\sqrt{\pi}\,\Gamma\big((n+1)/2\big)}{2\Gamma\big((n+2)/2\big)}$$ $$= \frac{(n-1)!!}{n!!} \cdot \begin{cases} \frac{\pi}{2} & for\ n = 2, 4, \ldots (even) \\ 1 & for\ n = 3, 5, \ldots (odd) \end{cases}$$ where • $\Gamma(n)$ is the gamma function • $n!!$ is the double factorial • $\int_0^{\pi/2} 1 \ dx = \pi/2$ • $\int_0^{\pi/2} \sin x \ dx = 1$ Optional enrichment. See Harold's Series Cheat Sheet for definitions. Source: Wolfram MathWorld, Wallis Cosine Formula

See Also

- Harold's AP Calculus BC Formulas Cheat Sheet – Formulas in rectangular, polar, and parametric
- Harold's Fundamental Theorem of Calculus Cheat Sheet
- Harold's Partial Fraction Expansion (Calculus) Cheat Sheet
- Harold's Taylor Series Cheat Sheet
- Harold's Infinite Series Cheat Sheet
- Harold's Series Cheat Sheet
- Harold's Series Convergence Tests Cheat Sheet

Acknowledgements

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