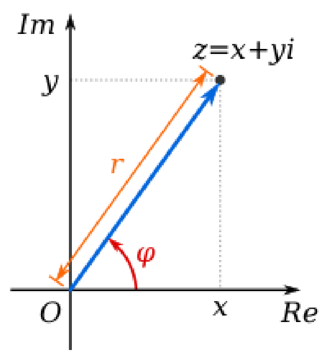
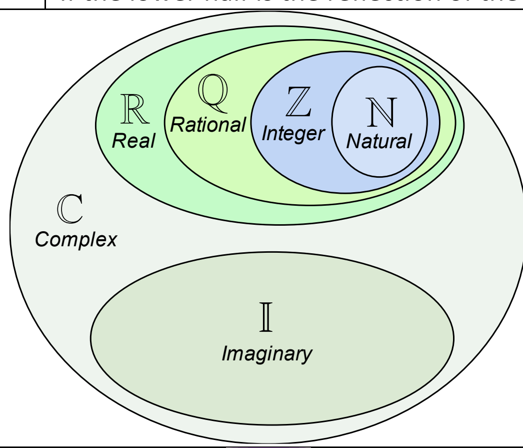
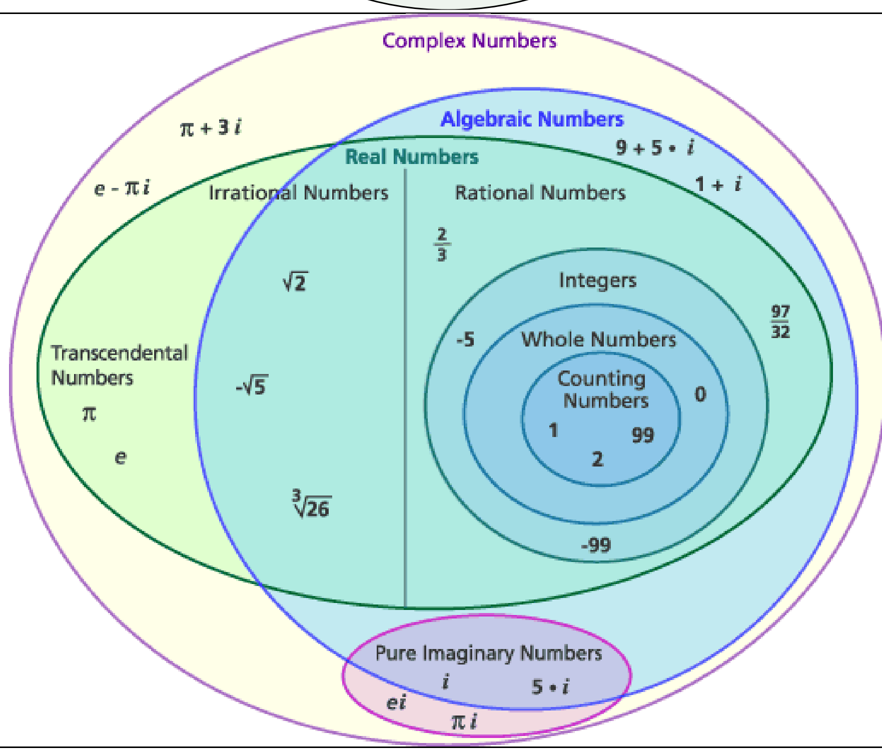
Harold's Complex Variables
Cheat Sheet
25 February 2025

Definitions

Name	Definition or Formula
Imaginary Number	$i = \sqrt{-1}$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$ i is used by mathematicians. j is used by electrical engineers.
Complex Number	 Rectangular Form (x, y): $z = x + iy$ $z = (x, y)$ where $x = \text{Re}\, z;\ y = \text{Im}\, z$ Polar Form (r, θ): $z = r(\cos\theta + i\sin\theta)$ Exponential Form (e^x): $\boldsymbol{z = re^{i\theta}}$ Parametric Form (ρ, θ): $z = z_0 + \rho e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ Shorthand: $e^z = \exp(z) = e^x e^{iy}$
Complex Conjugate	$\bar{z} = x - iy$ $\bar{z} = r(\cos\theta - i\sin\theta)$ $\bar{z} = re^{-i\theta}$
Modulus (Magnitude/Absolute Value)	$\lvert z \rvert = \sqrt{x^2 + y^2}$ $\lvert z \rvert = r$ $\lvert z \rvert^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2 = z \cdot \bar{z}$
Argument (Angle)	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ If $(-\pi < \theta \leq \pi)$ then principle value

Euler's Formula	$e^{i\theta} = \cos\theta + i\sin\theta$ Examples: $e^{i\frac{\pi}{2}} = i$ $e^{i\pi} = -1$ $e^{-i\frac{\pi}{2}} = -i$ $e^{i2\pi} = 1$
De Moivre's Formula	$z^n = [r(\cos\theta + i\sin\theta)]^n$ $\boldsymbol{z^n = r^n(\cos n\theta + i\sin n\theta)}$
Holomorphic Function (Analytic Function)	A complex variable function whose derivative exists at any point.
Meromorphic Function	A complex variable function that is holomorphic except in set points, which are poles.
Entire	A holomorphic function that is holomorphic $\forall z \in C$.
Reflection Principle	$\overline{f(z)} = f(\bar{z})$ If the lower half is the reflection of the upper half over the x-axis.





Algebraic Properties

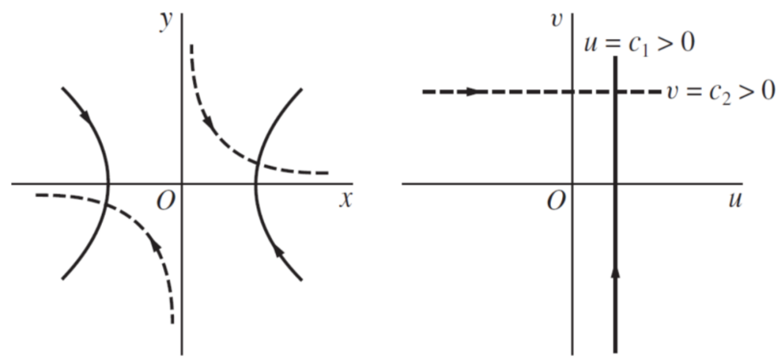
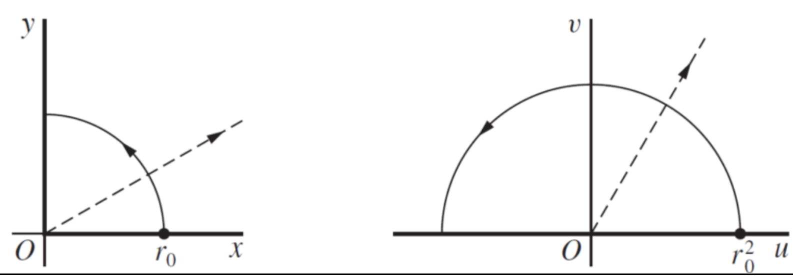
Property	Formula
Complex Numbers	$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ $z_1 - z_2 = (x_1 + x_2) - i(y_1 + y_2)$ $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ $\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}, z_2 \neq 0$ $\lvert z_1 \cdot z_2 \cdot z_3 \cdot \ldots \cdot z_n \rvert = \lvert z_1 \rvert \cdot \lvert z_2 \rvert \cdot \lvert z_3 \rvert \cdot \ldots \cdot \lvert z_n \rvert$ $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
Additive Inverses	$-z = (-x, -y)$ $-z = re^{i(\theta+\pi)}$
Multiplicative Inverses	$z^{-1} = \left(\frac{x}{x^2 + y^2}, i\frac{-y}{x^2 + y^2}\right), z \neq 0$ $z^{-1} = \frac{1}{r}e^{-i\theta}$
Complex Conjugates	$\lvert \bar{z} \rvert = \lvert z \rvert$ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$
Triangle Inequality	$\lvert z_1 \pm z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$ $\lvert z_1 \pm z_2 \rvert \geq \lvert z_1 \rvert - \lvert z_2 \rvert$ $\lvert z_1 + z_2 \rvert \geq \big\lvert \lvert z_1 \rvert - \lvert z_2 \rvert \big\rvert$
Exponentials	$z^n = r^n e^{in\theta}$ $z^c = e^{c \log z} = \exp(c \log z)$
Roots	$\sqrt[n]{z} = \sqrt[n]{r} \exp\left[i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$
Arguments (Angles)	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ $\arg(z_2^{-1}) = -\arg(z_2)$

Transcendental Properties

Property	Formula
Power	$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$ $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$ $e^z = e^{z+2\pi ki}$
Logarithms	$\log z = \ln\lvert z\rvert + i \arg z$ $\log e^z = z + 2n\pi i$ $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ $\ln\lvert z_1 z_2\rvert = \ln\lvert z_1\rvert + \ln\lvert z_2\rvert$ $\ln\lvert z_1 z_2\rvert + i \arg(z_1 z_2)$ $= \big(\ln\lvert z_1\rvert + i \arg(z_1)\big) + \big(\ln\lvert z_2\rvert + i \arg(z_2)\big)$
Trigonometric	$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \frac{e^{iz} + e^{-iz}}{2i}$ $\sin^2 z + \cos^2 z = 1$ $\tan^2 z + 1 = \sec^2 z$ $1 + \cot^2 z = \csc^2 z$ $\sin(2z) = 2 \sin z \cos z$ $\cos(2z) = \cos^2 z - \sin^2 z$ $\cos(2z) = 2\cos^2 z - 1$ $\cos(2z) = 1 - 2\sin^2 z$
Hyperbolic	$\sinh z = \frac{e^z - e^{-z}}{2}$ $\cosh z = \frac{e^z + e^{-z}}{2}$ $\sin(ix) = i \sinh x$ $\cos(ix) = \cosh x$ $\sin z = \sin x \cosh y + i \cos x \sinh y$ $\cos z = \cos x \cosh y - i \sin x \sinh y$ $\sinh z = \sinh x \cos y + i \cosh x \sin y$ $\cosh z = \cosh x \cos y - i \sinh x \sin y$

Inverse Trigonometric	$\sin^{-1} z = -i \ln\left[iz + \sqrt{1 - z^2}\right]$ $\cos^{-1} z = -i \ln\left[z + \sqrt{z^2 - 1}\right]$ $\tan^{-1} z = \frac{i}{2} \ln\left[\frac{i + z}{i - z}\right]$ $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\tan^{-1} z = \frac{1}{1 + z^2}$
Inverse Hyperbolic	$\sinh^{-1} z = \ln\left[z + \sqrt{1 + z^2}\right]$ $\cosh^{-1} z = \ln\left[z + \sqrt{z^2 - 1}\right]$ $\tanh^{-1} z = \frac{1}{2} \ln\left[\frac{1 + z}{1 - z}\right]$

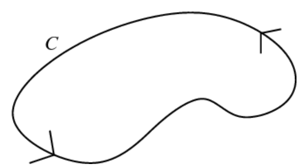
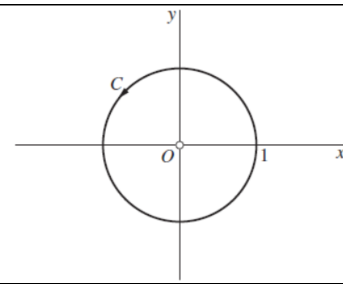
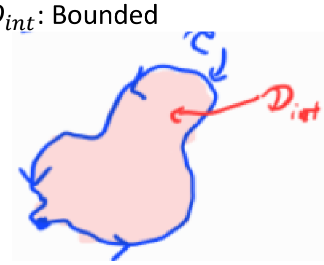
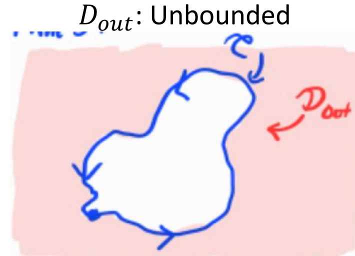
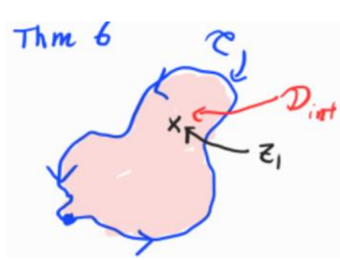
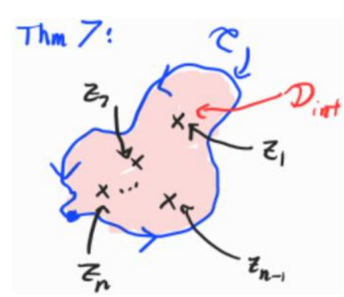
Functions

Name	Formula
Functions	$f(z) = f(x+iy) = u(x,y) + iv(x,y) = u + iv$ $f(z) = f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta) = u + iv$
Conic Mappings	Hyperbola (Rectangular Form): $w = z^2$ $u = x^2 + y^2 = c_1$ $v = 2xy = -2y\sqrt{y^2 + c_1}$  Circle (Polar Form): $w = z^2$ $w = r^2 e^{i2\theta}$ $\rho = r^2$ $\varphi = 2\theta$ 

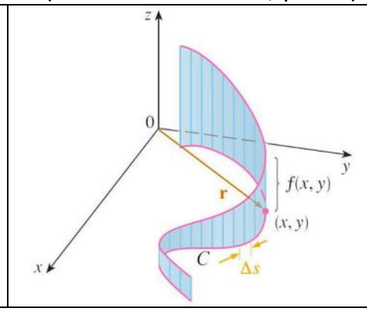
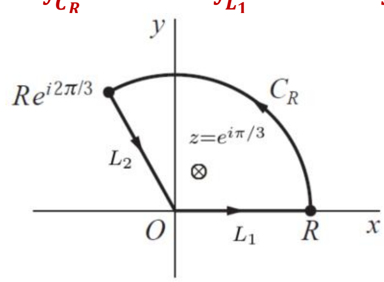
Differentiation

Name	Formula
Cauchy-Riemann Equations	Determines whether the given complex-valued function $f(z) = u + iv$ is analytic and differentiable. Rectangular Form: $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists at point $z_0 = x_0 + iy_0$ $u_x = v_y, \quad u_y = -v_x$ $f'(z_0) = u_x + iv_x$ where $u_x = \frac{\partial u}{\partial x}$ Polar Form: $f(z) = u(r, \theta) + iv(r, \theta)$ $ru_r = v_\theta, \quad u_\theta = -rv_r$ $f'(z_0) = e^{-i\theta}(u_r + iv_r)$
Laplace's Equation (Harmonic)	$H_{xx}(x, y) + H_{yy}(x, y) = 0$

Contours

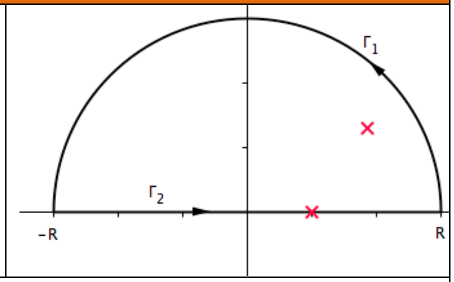
<table>
<tr><th>Name</th><th colspan="2">Definition</th></tr>
<tr><td>Simple Arc (C)
(Jordan arc)</td><td colspan="2">If arc C does not cross itself; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. E.g., open.</td></tr>
<tr><td>Contour (C)</td><td colspan="2">A closed path in the complex plane.
A piecewise smooth arc consisting of a finite number of smooth arcs joined end to end.</td></tr>
<tr><td rowspan="2">Simple Curve (C)</td><td>A simple arc where $z(b) = z(a)$. E.g., closed.</td><td>Simple closed curve
</td></tr>
<tr><td>Simple closed curve C defaults to a circle, $|z - 0| = r$, centered at 0 with radius r and interval $[0, 2\pi]$, oriented counterclockwise.
$C(r) = \{z \in \mathbb{C} : |z| = r\}$</td><td></td></tr>
<tr><td>Positively Oriented</td><td colspan="2">a simple closed curve, or a Jordan curve, is positively oriented when it is in the counterclockwise direction.</td></tr>
<tr><td>Simply Connected Domain</td><td colspan="2">D is a domain such that every simple closed contour within it encloses only points of D.</td></tr>
<tr><td>Branch Cut</td><td colspan="2">A portion of a line or curve that is introduced to define a branch F of a multiple-valued function f.</td></tr>
<tr><td>Regions Bound by Curve C</td><td>D_{int}: Bounded
</td><td>D_{out}: Unbounded
</td></tr>
<tr><td>Closed, Simple, Counter-Clockwise Oriented Curve</td><td>One Point, Simple Pole:
</td><td>Multiple Points, Simple Poles:
</td></tr>
</table>

Integration

Name	Formula
Complex Integration	$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C udx - vdy + i\int_C vdx - udy$ *"No corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane."* (Brown & Churchill, p.125)
Contour Integral (Complex $\mathbb{C}$)	Called a Line Integral if Real $\mathbb{R}$ numbers. $\boldsymbol{\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt}$ Change of Contour direction: $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$ 
Fundamental Theorem of Calculus with Contour Integral	$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)dt = F(z(t))]_a^b$ $= F(z(b)) - F(z(a)) = F(z_2) - F(z_1)$ Since $z(b) = z_2$ and $z(a) = z_1$. where z_0 is a point within the contour C.
Morera's Theorem	If $\int_C f(z)\,dz = 0$ then, f is **Holomorphic** over $\mathbb{R}$.
Cyclic Integral	$\oint_C f(z)\,dz$ Integral over a **closed** contour meaning the curve returns to its initial position ($a = b$). For circular contours, $\oint_{C_R} f(z)\,dz = \int_0^{2\pi} f(z)\,dz$ For non-circular contours, $\boldsymbol{\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz}$  Parameterize the arcs and identify the bounds of integration. $L_1: z = re^{i0}, \quad 0 \le r \le R$ $L_2: z = re^{i2\pi/3}, \quad R \le r \le 0$ $C_R: z = Re^{i\theta}, \quad 0 \le \theta \le 2\pi/3$

Cauchy-Goursat Theorem (Cauchy's Integral Theorem)	D_{int}: If C is closed, i.e., $z_0 = z_1$, then $$\oint_C f(z)\,dz = 0$$ D_{out}: Outside of closed C, at infinity (∞): If $$\lim_{z\to\infty} f(z)\,z = 0$$ Then $$\oint_C f(z)\,dz = 0$$
Cauchy Integral Formula	Turns a contour integral into a derivative. Simple: $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$ General: $$\boldsymbol{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz}$$ where $n = 0, 1, 2, \ldots;\ 0! = 1;\ f^{(0)}(z_0) = f(z_0)$.
Jordan's Lemma	Estimation Lemma: $$\left\|\int_C f(z)\,dz\right\| \le \text{length}(C)\cdot\max_{z\in C}\|f(z)\|$$ Common Application: If $$\lim_{z\to\infty} f(z)\,z = 0$$ Then $$\int_{-\infty}^{\infty} f(at+b)\,a\,dt\,x = \int_L f(z)\,dz = \lim_{R\to\infty}\oint_{C_R} f(z)\,dz$$ where line $L = \{at + b\}, -\infty < t < \infty$ and $R =$ semi-circle radius along this line.

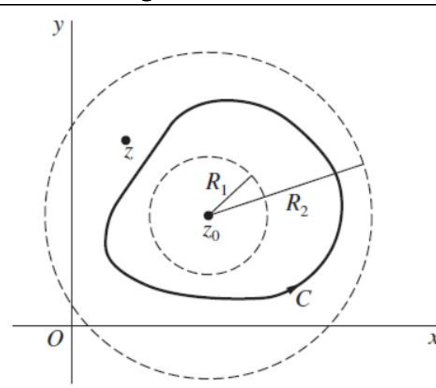
Poles and Residues

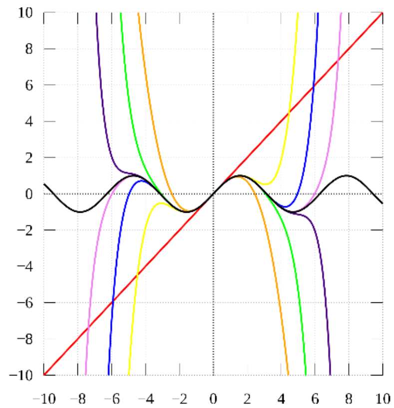
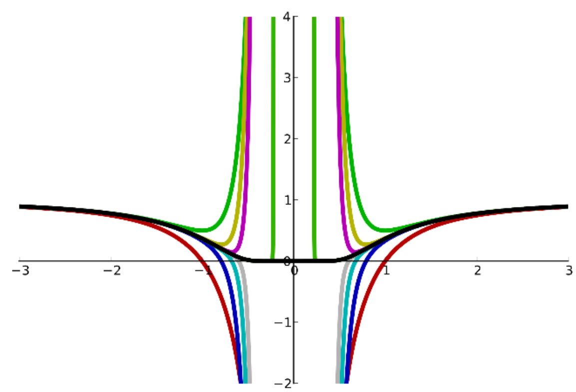
<table>
<tr><th>Name</th><th colspan="2">Formula</th></tr>
<tr><td rowspan="2">Poles</td><td>Roots in the denominator of a complex function that is holomorphic (complex differential). E.g., Singularity, vertical asymptote.</td><td></td></tr>
<tr><td colspan="2">Poles are zeros in the denominator of $f(z)$.
$$f(z) = \frac{\varphi(z)}{(z-z_0)^m}$$
<u>Simple Pole</u>:
$$z_0 \text{ is a pole of order } 1\ (m = 1).$$
<u>High-Order Pole</u>:
$$z_0 \text{ is a pole of order } m\ (m = 2, 3, 4, \dots).$$
Theorem: If z_0 is a pole of function f, then
$$\lim_{z \to z_0} f(z) = \infty.$$</td></tr>
<tr><td>Residue</td><td colspan="2">$$\mathbf{Res}_{z_1}(f) = \lim_{z \to z_1}(z - z_1)\, f(z)$$
Observation: Since $z_1$is a simple pole, then the $\mathrm{Res}_{z_1}(f)$ turns the function f into a function with a hole or hollow point. The limit makes the remaining function appear continuous. “Remove the pole and cover the hole.”

Tip: If Laurent Series centered at 0, then
$Res_{z_1}(f) = a$ the z^{-1} term of $f(z) = \cdots + az^{-1} + \cdots$.</td></tr>
<tr><td>Cauchy’s Residue Theorem
(Simple Poles)</td><td colspan="2"><u>One</u> Point, Simple Pole inside Contour C:
If exists
$$\mathrm{Res}_{z_1}(f) = \lim_{z \to z_1}(z - z_1)\, f(z)$$
Then
$$\oint_C f(z)\, dz = 2\pi i\, \mathrm{Res}_{z_1} f(z)$$</td></tr>
</table>

	Multiple Points, Simple Poles inside Contour C: If these exist $\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) =$ $Res_{z_1}(f) = \lim_{z \to z_1} (z - z_1) f(z)$ $+ \dots +$ $Res_{z_n}(f) = \lim_{z \to z_n} (z - z_n) f(z)$ Then $\boldsymbol{\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)}$
	Special Case: If $f(z)$ is even, then $\int_{-\infty}^{\infty} f(z)\, dz = \frac{1}{2} \int_{0}^{\infty} f(z)\, dz$
Residue of High-Order Poles	General: Works with higher-order poles. If $f(z) = \frac{\varphi(z)}{(z - z_0)^m} \quad (m = 1, 2, 3, \dots)$ where $\varphi(z)$ is analytic and nonzero at z_0, then $\boldsymbol{\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \quad (m = 1, 2, \dots)}$ Simple: $\operatorname*{Res}_{z=z_0} f(z) = \varphi(z_0) \text{ when } m = 1$ $\text{since } \varphi^{(0)}(z_0) = \varphi(z_0) \text{ and } 0! = 1.$ Tip: $\text{If } f(z) = \frac{g(z)}{(z - z_0)^m (z - z_1)^p \cdots (z - z_{n-1})^q}$ $\text{then choose } \varphi(z) = \frac{g(z)}{(z - z_1)^p \cdots (z - z_{n-1})^q}.$
Residue of Simple Poles (shortcut)	**Theorem**: Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0,$ $q(z_0) = 0, \text{and}$ $q'(z_0) \neq 0,$ then z_0 is a simple **pole** of the quotient $p(z)/q(z)$ and $\boldsymbol{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}.$
Zeros and Poles	**Theorem**: Suppose that: (a) two functions p and q are analytic at a point z_0; (b) $p(z_0) \neq 0$ and q has a **zero** of order m at z_0. Then the quotient $p(z)/q(z)$ has a **pole** of order m at z_0.

Series

Name	Formula
Liouville's Theorem	If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.
Fundamental Theorem of Algebra	$P(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0, \quad (a_n \neq 0, n \geq 1)$ $P(z) = c(z - z_n) \ldots (z - z_2)(z - z_1)$ Any polynomial of degree n has at least one zero in the complex plane. That is, there exists at least one point z_0 such that $P(z_0) = 0$.
Maximum Modulus Principle	**Theorem:** If a function f is analytic and not constant in a given domain D, then the modulus $\lvert f(z)\rvert$ has no maximum value in D. That is, there is no point z_0 in the domain such that $\lvert f(z)\rvert \leq \lvert f(z_0)\rvert$ for all points z in it.
	Corollary: Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $\lvert f(z)\rvert$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.
Complex Variable Convergence	$\lim_{n\to\infty} z_n = z$ $\text{iff } \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$ $\text{where } z_n = x_n + iy_n$
Complex Series Convergence	$\sum_{n=1}^{\infty} z_n = S$ $\text{iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$ $\text{where } S = X + iY$
Series Convergence	**Corollary 1**: If a series of complex numbers converges, the n[th] term converges to zero as n tends to infinity. **Corollary 2:** The absolute convergence of a series of complex numbers implies the convergence of that series.
Annular Domain $R_1 < \lvert z - z_0\rvert < R_2$	
Transcendental Series	See Harold's Taylor Series Cheat Sheet for a comprehensive list of the Maclaurin series of all transcendental functions.

Taylor Series	$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n \quad \text{where disk } (\lvert z - z_0 \rvert < R_0)$$ $$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{where } (n = 0, 1, 2, \dots)$$ Series converges to $f(z)$ when z lies in the stated open disk. If $z_0 = 0$, then **Maclaurin series**. 
Laurent Series	$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$ $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \ \text{ where } (n = 0, 1, 2, \dots)$$ $$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \ \text{ where } (n = 1, 2, 3, \dots)$$ $$\text{where } (R_1 < \lvert z - z_0 \rvert < R_2)$$  Taylor Series Form: $$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$ $$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!} \ \text{ where } (n = 0, 1, 2, \dots)$$ $$\text{and } (R_1 < \lvert z - z_0 \rvert < R_2)$$ If no poles, then **Taylor series**.

Power Series

Name	Formula
Power Series	$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$ where $(\lvert z - z_0\rvert < R)$
Absolute and Uniform Convergence	**Theorem 1**: If a power series converges when $z = z_1\ (z_1 \neq z_0)$, then it is **absolutely** convergent at each point z in the open disk $\lvert z - z_0\rvert < R_1$ where $R_1 = \lvert z_1 - z_0\rvert$.
	Theorem 2: If z_1 is a point inside the circle of convergence $\lvert z - z_0\rvert = R$ of a power series then that series must be **uniformly** convergent in the closed disk $\lvert z - z_0\rvert \leq R_1$ where $R_1 = \lvert z_1 - z_0\rvert$.
Continuity of Sums	**Theorem**: A power series represents a **continuous** function $S(z)$ at each point inside its circle of convergence $\lvert z - z_0\rvert = R$.
Integration	**Theorem**: Let C denote any contour interior to the circle of convergence of the power series and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be **integrated** term by term over C; that is, $$\int_C g(z)\, S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)\, (z - z_0)^n dz\,.$$
Differentiation	**Theorem**: The power series can be **differentiated** term by term. That is, at each point z interior to the circle of convergence of that series, $$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$
Leibniz's Rule for the n[th] Derivative	$$[f(z)\, g(z)]^{(n)} = \sum_{n=0}^{\infty} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$$ where $(n = 1, 2, \ldots)$ $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$ where $(k = 0, 1, 2, \ldots, n)$
Uniqueness Representations	**Theorem 1**: If a power series converges to $f(z)$ at all points interior to some circle $\lvert z - z_0\rvert = R$, then it is the **Taylor series** expansion for f in powers of $z - z_0$.
	Theorem 2: If a series $$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the **Laurent series** expansion for f in powers of $z - z_0$ for that domain.
Multiplication	Let $f(z), g(z), h(z),$ and $k(z)$ all be different power series.
	$f(z)g(z) = h(z)$
Division	$$\frac{f(z)}{g(z)} = k(z)$$

College Course

- **Course**: NYU MATH-UY-4434: Applied Complex Variables, 2024.
- **Textbook**: Complex Variables and Applications, 9th Edition, Chapters 1-7, James Ward Brown & Ruel V. Churchill, McGraw-Hill Education, 2014.

References

- Bhoris Dhanjal, Bhorice2099 on Reddit (2021, 11 October). Imgur, Introductory Complex Analysis Cheat Sheet v2.

Introductory Complex Analysis Cheat Sheet

Field of Complex Numbers

We construct the field of complex numbers as the following quotient ring, $\mathbb{C} = \mathbb{R}[x]/\langle x^2+1\rangle$

Algebra of Complex Numbers

- Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$
- Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$
- Division: $\dfrac{a+ib}{c+id} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$
- Square root: $\sqrt{a+ib} = \pm\left(\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} + i\dfrac{b}{|b|}\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}\right)$
- $\Re(a+ib) = a, \Im(a+ib) = b$

Conjugation, Absolute Value

- **Complex conjugation:** $\overline{a+ib} = a-ib$
  - $\overline{a+b} = \overline{a}+\overline{b}$
  - $\overline{ab} = \overline{a}\cdot\overline{b}$

  Geometrically, conjugation is reflection over the real axis.
- **Absolute value:** $|a| = +\sqrt{a\overline{a}}$
  - $|ab| = |a|\cdot|b|$
  - $|a+b|^2 = |a|^2+|b|^2+2\Re(a\overline{b})$
  - $|a-b|^2 = |a|^2+|b|^2-2\Re(a\overline{b})$
  - $|a+b|^2+|a-b|^2 = 2(|a|^2+|b|^2)$

  The absolute value function forms the metric on $\mathbb{C}$. $\mathbb{C}$ is complete under this metric.

Basic Topological definitions in $\mathbb{C}$

Some basic results:

- For $z_0 \in \mathbb{C}, r > 0$ we denote the ball (i.e. disk) of radius r around z_0 to be $B(z_0, r) = \{z \in \mathbb{C} \mid |z-z_0| < r\}$
- A point $z \in \mathbb{C}$ is a **limit point** of $E \subseteq \mathbb{C}$ if $\forall\varepsilon > 0$, $B(z,\varepsilon)\cap E$ contains a point other than z.
- A subset $E \subseteq \mathbb{C}$ is said to be **open** if $\forall z \in E, \exists\, r > 0$, s.t. $B(z,r) \subset E$.
- A subset $E \subseteq \mathbb{C}$ is said to be **closed**, if $\mathbb{C}\setminus E$ is open in C. Or equivalently a set which contains all its limit points.

Some properties of open sets:

- $\mathbb{C}$ and $\varnothing$ are open subsets of $\mathbb{C}$.
- All finite intersections of open sets are open sets.
- The collection of all open sets on $\mathbb{C}$ form a topology on $\mathbb{C}$.

Interior, closure, density

- **Interior:** Let $E \subseteq \mathbb{C}$. The interior of E is defined as, E°=set of all interior points of E, or equivalently, $\cup\{\Omega \mid \Omega \subseteq E \wedge \Omega \text{ is open in } \mathbb{C}\}$
- **Closure:** Let $E \subseteq \mathbb{C}$. The closure of E is defined as $\bigcap\{F \mid E \subseteq F \wedge F \text{ is closed in } \mathbb{C}\}$
- **Density:** Let $E \subseteq D$, the closure of E in D is D. Then E is called dense in D.

Path : A path in a metric space from a point $x \in X$ to $y \in Y$ is a continuous mapping $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

Separated and Connected

For a metric space (X, d).

- **Separated:** X is separated if $\exists$ disjoint non-empty open subsets A, B of X s.t. $X = A \cup B$.
- **Connected:**
  - X is connected if it has no separation.
  - X is connected $\iff$ X does not contain a proper subset of X which is both open and closed in X.
  - Continuous functions preserve connectedness.
  - An open subset $\Omega \in \mathbb{C}$ is connected $\iff$ for $z, w \in \Omega$, there exists a path from z to w.

Basic Topological definitions in C contd.

Open cover: Let (X,d) be a metric space and E be a collection of open sets in X. We say that $\mathscr{U}$ is an open cover of a subset $K \subseteq X$, if $K \subset \bigcup\{\mathscr{U} \mid \mathscr{U} \in E\}$

Compactness: For some $K \subseteq X$ is compact if for every open cover E of K, there exists $E_1, \cdots .E_n \in E$ s.t. $K \subset U_{i=1}^{n} E_n$, i.e. it is compact if it has a finite open cover.

- In a metric space, a compact set is closed.
- A closed subset of a compact set is closed.

Limit point compact: We say a metric space X is limit point compact if every infinite subset of X has a limit point.

- If X is a compact metric space, then it is also limit point compact.

Sequentially compact: We say a metric space X is sequentially compact if every sequence has a convergent sub-sequence.

- If X is limit point compact then X is sequentially compact.
- Let X be sequentially compact, then X is a compact metric space.

Lebesgue number lemma: Let X be sequentially compact, and let $\mathscr{U}$ be an open cover of X. Then $\exists\,\delta > 0$ s.t. for $x \in X$, $\exists\, u \in \mathscr{U}$ s.t. $B(x,\delta) \subseteq u$.

Isometries on the Complex Plane

A function $f : \mathbb{C} \to \mathbb{C}$ is called an **isometry** if $|f(z)-f(w)| = |z-w|, \forall z, w \in \mathbb{C}$.

- Let f be an isometry s.t. $f(0) = 0$, then the inner product $\langle f(z), f(w)\rangle = \langle z, w\rangle, \forall z, w \in \mathbb{C}$.
- If f is an isometry s.t. $f(0) = 0$ then f is a linear map.
- The standard argument for $a+ib \in \mathbb{C}$, $\mathrm{Arg}(a+ib) = \tan^{-1}\frac{b}{a}$

Functions on the Complex Plane

Uniform convergence: Let $\Omega \subseteq \mathbb{C}$ and $f_1, \cdots, f_n : \Omega \to \mathbb{C}$ be a set of functions on Ω. We say, $\{f\}_{n\in\mathbb{N}}$ converges uniformly to f if given $\varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon, \forall x \in \Omega$ and $n \geq N$.

Complex exponential: For $z \in C$, $\exp(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}$

Trigonometric functions: For $z \in \mathbb{C}$, $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(x) = \frac{e^{iz}-e^{-iz}}{2}$

Hyperbolic trigonometric functions: For $z \in \mathbb{C}$, $\cosh(x) = \frac{e^{z}+e^{-z}}{2}$ and $\sinh(z) = \frac{e^{z}-e^{-z}}{2}$

Complex differentiability

Complex derivative: Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$, we say that f is complex differentiable at a point $z_0 \in \Omega$ if z_0 is an interior point and the following limit exists $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$. The limit is denoted as $f'(z_0)$ or $\frac{df(z)}{dz}$.

Holomorphic functions: If $f : \Omega \to \mathbb{C}$ is complex differentiable at every point $z \in \Omega$, then f is said to be a holomorphic on Ω. **Entire function:** Functions which are complex differentiable on $\mathbb{C}$ are called entire functions.

- Complex differentiability implies continuity.
- Complex derivatives of a function are linear transformations.
- **Product rule:** If $f, g : \Omega \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$. Then fg is complex differentiable at z_0 with derivative $f'(z_0)g(z_0) + g'(z_0)f(z_0)$.
- **Quotient rule:** If $f, g : \Omega \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, and g doesn't vanish at z_0. Then $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0)-g'(z_0)f(z_0)}{g(z_0)^2}$
- **Chain rule:** If $f : \Omega \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, and $f(\Omega) \subseteq D$. Then $g(f(x))'(z_0) = g'(f(z_0))f'(z_0)$

Power Series

Formal Power Series: A formal power series around $z_0 \in \mathbb{C}$ is a formal expansion $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, where $a_n \in \mathbb{C}$ and z is indeterminate.

Radius of convergence: For a formal power series $\sum a_n(z-z_0)^n$ the radius of convergence $R \in [0,\infty]$ given by $R = \liminf_{n\to\infty}|a_n|^{-1/n}$. Using the ratio test is identical i.e. $R = \liminf_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}$.

- The series converges absolutely when $z \in B(z_0, R)$, and for $r < R$, the series converges uniformly, else if $|z-z_0| > R$ the series diverges.
- Let $z \in \mathbb{C}$ s.t. $|z-z_0| > R$, then $\exists$ infinitely many $n \in N$ s.t. $|a_n|^{-1/n} < |z-z_0|$.

Abel's Theorem: Let $F(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with a positive radius of convergence R, suppose $z_1 = z_0 + Re^{i\theta}$ be a point s.t. $F(z_1)$ converges. Then $\lim_{r\to R^-} F(z_0 + re^{i\theta}) = F(z_1)$

Differentiation of Power Series

Let $F(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series around z_0 with a radius of convergence R. Then F is **holomorphic** in $B(z_0, R)$.

- $F(x)' = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ with same radius of convergence R.
- $a_n = \frac{F^n(z_0)}{n!}$

Cauchy product of two power series: For power series $F(z) = \sum a_n(z-z_0)^n$ and $G(z) = \sum a_n(z-z_0)^n$ with degree of convergence at least R. Then the Cauchy product $F(z)G(z) = \sum c_n(z-z_0)^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ also has degree of convergence at least R.

Cauchy-Riemann Differential Equations

For a complex function $f(z) = u(z) + iv(z)$,
$f'(x) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ or $f'(x) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Therefore, we get the two **Cauchy-Riemann Differential equations,**

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$
- $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

A function is holomorphic **iff** it satisfies the Cauchy-Riemann equations.

Wirtinger derivatives:

- $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$
- $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$

If f is holomorphic at z_0 then, $\frac{\partial f}{\partial \overline{z}} = 0$ and $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$

Harmonic Functions

Laplacian: Define $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Harmonic function: Let $u : \Omega \to \mathbb{R}$ be a twice differentiable function. We say that u is a harmonic function if $\Delta u = 0$

For any holomorphic function f, $\Re(f), \Im(f)$ are examples of harmonic functions, but there are harmonic functions which are not holomorphic.

Boundary of a set: For a metric space $X, \Omega \in X$,
the boundary of $\Omega = \partial\Omega = \overline{\Omega} \cap \overline{\Omega^C}$

Maximum principle for harmonic functions: Let $u : \Omega \to \mathbb{R}$ be a twice differentiable harmonic function. Let $k \subset \Omega$ be a compact sub set of Ω. Then, $\sup_{z\in k} u(z) = \sup_{z\in\partial k} u(z)$ and $\inf_{z\in k} u(z) = \inf_{z\in\partial k} u(z)$

Maximum principle for holomorphic functions: Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, for compact $k \subseteq \Omega$, we have, $\sup_{z\in k}|f(z)| = \sup_{\partial k}|f(z)|$

Harmonic conjugate: Let $u : \Omega \to \mathbb{R}$ be a twice differentiable harmonic function. We say that $v : \Omega \to \mathbb{R}$ is a harmonic conjugate of u if $f = u + iv$ is holomorphic.

• For a harmonic function from $\mathbb{C}$ to $\mathbb{R}$ there exists a uniquely determined harmonic conjugate from $\mathbb{C}$ to $\mathbb{R}$ (up to constants).

Riemann Sphere

Extended complex plane: $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$
Consider S^2, associate every point $z = x + iy$ with a line L that connects to the point $P = (0,0,1)$. $L = (1-t)z + tP$, where $t \in \mathbb{R}$.
The point at which L for some z touches S^2 is given as $\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$, associate P with ∞. This gives a stereographic projection of the complex plane unto S^2. This sphere is known as the Riemann sphere.

Möbius transformations

A map $S(z) = \dfrac{az+b}{cz+d}$ for $a,b,c,d \in \mathbb{C}$ is called a Möbius transformation if $ad - bc \neq 0$.
Every mobius transformation is holomorphic at $\mathbb{C} \setminus \{-d/c\}$, i.e. every point other than is zero.

- The set of all mobius transformations is a group under transposition.
- S forms a bijection with $\widehat{\mathbb{C}}$

Every mobius transformation can be written as composition of,

1. Translation: $S(z) = z + b, b \in \mathbb{C}$
2. Dilation: $S(z) = az, a \neq 0, a = e^{i\theta}$
3. Inversion: $S(z) = 1/z$

Curves in $\mathbb{C}$

A continuous parametrized curve is a continuous map $\gamma : [a,b] \to \mathbb{C}$ for $a, b \in \mathbb{R}$.

- If $a = b$ the curve is trivial.
- $\gamma(a)$ is initial point and $\gamma(b)$ is terminal point.
- γ is said to be closed if $\gamma(a) = \gamma(b)$.
- γ is said to be simple if it is injective, i.e. doesn't "cross" itself.
- A curve $-\gamma$ is a reversal of γ if $\gamma : [-a,-b] \to \mathbb{C}$ and if $-\gamma(t) = \gamma(-t)$
- γ is said to be continuously differentiable if $\gamma'(t_0)$ (defined usually) exists and is continuous.

Reparametrization: We say a curve $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ is a continuous reparametrization of $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, if there exists a homeomorphism $\varphi : [a_1, b_1] \to [a_2, b_2]$ s.t.$\varphi(a_1) = a_2, \varphi(b_1) = b_2$ and $\gamma_2(\varphi(t)) = \gamma_1(t) \forall t \in [a_1, b_1]$.

- Reparametrization is an equivalence relation.

Arc length: Arc length of curve $\gamma = |\gamma| = \sup \sum_{i=0}^{n} |\gamma(x_{i+1} - \gamma(x_i))|$ for all partitions of $[a,b]$.

- A curve that has a finite arc length is called **rectifiable**.
- $|\gamma| = \displaystyle\int_a^b |\gamma'(t)|\, dt$

First Fundamental Theorem of Calculus

Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $F : \Omega \to \mathbb{C}$ be called the anti-derivative of f, i.e. F is holomorphic in Ω and $F'(z) = f(z), \forall z \in \Omega$. For a rectifiable curve γ, $\int_\gamma f(z)dz = F(z_1) - F(z_0)$, where z_0 is the initial point and z_1 is the terminal point.

Second Fundamental Theorem of Calculus

Let $f : \Omega \to \mathbb{C}$ be a continuous function such that $\int_\gamma f = 0$. Whenever γ is a closed polygonal path contained in Ω. For fixed $z_0 \in \Omega$, define a path γ_1 from z_0 to z_1 such that $F(z_1) = \int_{\gamma_1} f(z)\, dz$. Then F is a well defined holomorphic function s.t. $F'(z_1) = f(z_1)\ \forall z_1 \in \Omega$

Properties of complex integration

For continuously differentiable curves $\gamma : [a,b] \to \mathbb{C}$, and $\sigma : [b,c] \to \mathbb{C}$

- For a reparametrization $\widetilde{\gamma}$ of γ we can say that $\int_\gamma f(z)\, dz = \int_{\widetilde{\gamma}} f(z)\, dz$
- $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$
- $\int_{\gamma+\sigma} f(z)\, dz = \int_\gamma f(z)\, dz + \int_\sigma f(z)\, dz$
- $\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$
- If f is bounded by M then $\int_\gamma f(z)\, dz \leq M|\gamma|$
- For $c \in \mathbb{C}$, we have, $\int_\gamma (cf + g)(z)\, dz = c\int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz$

Homotopy of curves

Consider two curves $\gamma_0, \gamma_1 \to \Omega$ with the same initial and end point $[a,b]$.
We say that γ_0 is homotopic to γ_1 ($\gamma_0 \sim \gamma_1$) if there exists a continuous map $H : [0,1] \times [a,b] \to \Omega$ s.t. $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$, $\forall t \in [a,b]$. $H(s,a) = z_0, H(s,b) = z_1\ \forall s \in [0,1]$
For **closed curves** γ_0 at z_0 and γ_1 at z_1, we say that γ_0 is homotopic to γ_1 as closed curves if there exists a continuous map $H : [0,1] \times [a,b] \to \Omega$, s.t. $H(0,t) = \gamma_0(t), H(1,t) = \gamma_1(t)$, $\forall t \in [a,b]$. And $H(s,a) = H(s,b)$, $\forall s \in [0,1]$.

- Homotopy is an equivalence relation.

Cauchy-Goursat Theorem

Cauchy-Goursat theorem: If a curve γ_0 is homotopic to a reparametrization of γ_1 then, the integral of some function $f : \Omega \to \mathbb{C}$ is homotopy invariant, i.e., $\displaystyle\int_{\gamma_0} f = \int_{\gamma_1} f$
Alternative statement: Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, and $\gamma_0 : [a,b] \to \Omega$ is a rectifiable curve which is null-homotopic (i.e. homotopic to a constant map). Then, $\displaystyle\int_{\gamma_0} f(z)\, dz = 0$

Cauchy's theorem for convex domains

Let $\Omega \subseteq \mathbb{C}$ be a convex and open set and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then f has an anti derivative F on Ω, and if γ is a closed rectifiable curve on Ω then $\int_\gamma f = 0$.

Cauchy's integral formula

Let $f : \Omega \to \mathbb{C}$ be holomorphic. Fix $z_0 \in \Omega$ and let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$. Suppose γ is a closed curve in $\Omega \setminus \{z_0\}$ s.t. γ is homotopic to a reparametrization to γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

Complex analytic function

An alternative statement, we say $f : \Omega \to \mathbb{C}$ is complex analytical if given $z_0 \in \Omega, \exists B(z_0, r) \subseteq \Omega$ s.t. the formal power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to f.
Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Suppose for $z_0 \in \Omega, \overline{B(z_0, r)} \subset \Omega$, then for every $n \in \mathbb{N}$, let $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ where $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to $f(z)$.
Corollary: If $f : \Omega \to \mathbb{C}$ is holomorphic then f' is also holomorphic. Therefore f is infinitely differentiable.

Factor theorem for analytic function

For a analytic function $f : \Omega \to \mathbb{C}$ s.t. $f(z_0) = 0$ at $z_0 \in \Omega$, $\exists$ a unique analytic function $g : \Omega \to \mathbb{C}$ s.t. $f(z) = (z - z_0)g(z)$

Principle of analytical continuation

- Let Ω be open and connected subset of $\mathbb{C}$. and $f, g : \Omega \to \mathbb{C}$ be analytic functions on Ω. Suppose f, g agree on a non-empty subset of Ω, and this subset has an accumulation point. Then $f \equiv g$ on Ω.
- A consequence to this is that, non-trivial holomorphic functions have isolated zeros.

Higher-order Cauchy integral formula

Let $f : \Omega \to \mathbb{C}$ be analytic on Ω and $z_0 \in \Omega$ with $\overline{B(z_0, r)} \subseteq \Omega$. Let γ be a closed curve in $\Omega \setminus \{z_0\}$ that is homotopic to a reparametrization of γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Cauchy estimates: If $|f(z)| \leq M\ \forall z \in \gamma([0, 2\pi])$ then, $\forall n \in \mathbb{N}$, then we have $|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$

Liouville's Theorem

Let f be a entire function which is bounded. Then f is a constant function.

Fundamental Theorem of Algebra

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a non constant polynomial s.t. $a_i \in \mathbb{C}, a_n \neq 0$. Then $\exists z_1, z_2, \ldots, z_n$ s.t. $p(z) = a_n(z - z_1)\ldots(z - z_n)$.

Morera's Theorem

Let $f : \Omega \to \mathbb{C}$ be a continuous function such that, $\int_\gamma f(z)\, dz = 0, \forall$ closed polygonal paths $\gamma \in \Omega$. Then f is holomorphic on Ω.

Uniform limit of holomorphic functions

Let $f_n : \Omega \to \mathbb{C}$ be a holomorphic on $\Omega, \forall n \in \mathbb{N}$ s.t. f_n converges uniformly on compact sets to f. Then f is holomorphic.

Winding number

Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve and let z_0 be a point not in the image of γ. Then the winding number of γ around z_0 is

$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$$

- Winding number is invariant over homotopy.
- Let z_0 be a point not in the image of γ then $\exists r > 0$ s.t. for $z \in B(z_0, r), W_\gamma(z_0) = W_\gamma(z)$
- The winding number is always an integer.
- The winding number is locally constant.

Generalized Cauchy Integral formula: Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and $\gamma : [a,b] \to \Omega$ be a closed curve which is null homotopic. Then for z_0 not in the image of γ,

$$f(z_0)W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)}\, dz$$

Open Mapping Theorem

- $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then $G : \Omega \times \Omega \to \mathbb{C}$ given by

$$G(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{when } z \neq w \\ f'(z) & \text{when } z = w \end{cases}$$

  then G is continuous.
- Let $f : \Omega \to \mathbb{C}$ be holomorphic on some open set. Suppose $z_0 \in \Omega$ s.t. $f'(z_0) \neq 0$. Then $\exists$ a neighbourhood U of $z_0 \in \Omega$ s.t. f restricted to U is injective. And $V = f(U)$ is an open set and the inverse $g : V \to U$ of f is holomorphic.
- Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on open, connected set Ω. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. Then $\exists$ a neighbourhood U of z_0 and bijective holomorphic function φ on U s.t. $f(z) = w_0 + (\varphi(z))^m$ for $z \in U$ and some integer $m > 0$. And φ maps U unto $B(0,r)$ for some $r > 0$.

Open Mapping Theorem: Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on open connected set Ω, then $f(\Omega)$ is an open set.

Schwarz reflection principle

Let Ω be a open connected set which is symmetric w.r.t $\mathbb{R}$. Then define the following,

- $\Omega_+ = \{z \in \Omega \mid \Im(z) > 0\}$
- $\Omega_- = \{z \in \Omega \mid \Im(z) < 0\}$
- $I = \{z \in \Omega \mid \Im(z) = 0\}$

Schwarz reflection principle: Let Ω be defined as above. Then if $f : \Omega_+ \bigcup I \to \mathbb{C}$ which is continuous on $\Omega_+ \bigcup I$ and holomorphic on Ω_+. Suppose for $f(x) \in \mathbb{R}$, $\forall x \in I$ then there exists $g : \Omega \to \mathbb{C}$ holomorphic on Ω s.t. $g(z) = f(z)$ for $z \in \Omega_+ \bigcup I$

Singularity of a holomorphic function

- **Isolated singularity:** If f is holomorphic on $B(z_0, R) \setminus \{z_0\}$ for some $R > 0$ then z_0 is called an isolated singularity.
- **Removable singularity:** Let z_0 be an isolated singularity of a holomorphic function f as defined above. It is called removable if there exists holomorphic function g on $B(z_0, R)$ s.t. $g(z) = f(z)$ on $B(z_0, R) \setminus \{z_0\}$.
- **Riemann removable singularity theorem:** Let z_0 be an isolated singularity of a function f, then z_0 is a removable singularity if and only if f is locally bounded around z_0.
- **Pole:** If z_0 is an isolated singularity as defined above and if $\lim_{z\to z_0} |f(z)| = \infty$ then z_0 is called a pole of f.
- **Essential singularity:** A singularity that is neither removable nor a pole.

Doubly infinite series

Let z_n be a function defined for $n = 0, \pm 1, \pm 2, \cdots$, then it is doubly infinite.

- A doubly infinite series converges if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ both converge.
- Splitting up the series in similar manners you can define absolute and uniform convergence.

Annulus

An annulus $A(z_0, R_1, R_2)$ around a point z_0 for $0 \leq R_1 \leq R_2$ is the set of all $z \in \mathbb{C}$ s.t. $R_1 \leq |z - z_0| \leq R_2$.

Laurent series expansion

Let f be a function holomorphic on $A(z_0, R_1, R_2)$, then there exists $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ s.t.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the doubly infinite series converges absolutely and uniformly in some $A(z_0, r_1, r_2)$ when $R_1 < r_1 < r_2 < R_2$.

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where $\gamma(z) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $R_1 < r < R_2$.

Important results

- f has a removable singularity at $z_0 \iff a_n = 0$ for $n < 0$ in the Laurent series expansion of f
- f has a pole at z_0 of order $m \iff a_n = 0$ for $n < -m$ in the Laurent series expansion of f.
- f has a essential singularity at $z_0 \iff a_n \neq 0$ for infinitely many negative integers n.

Casorati–Weierstrass theorem

Let z_0 be an essential singularity of f then given $\alpha \in \mathbb{C}$, there exists a sequence $z_n \in B(z_0, R) \setminus \{z_0\}$ s.t. $z_n \to z_0$ and $f(z_n) \to \alpha$.

- Alternatively, f approaches any given value arbitrarily closely in any neighborhood of an essential singularity.

Meromorphic functions

Let Ω be a open connected subset of $\mathbb{C}$ and let $S \subset \Omega$. Let $f : \Omega \setminus S \to \mathbb{C}$ be holomorphic on Ω. We say that f is a meromorphic function on Ω if,

- S is a discrete set.
- f either has removable singularities or poles at point of S.

Operations on meromorphic functions

Let $\mathcal{M}(\Omega)$ denote the equivalence classes of meromorphic functions over Ω.

- We say that two meromorphic functions $f : \Omega \setminus S_1$ and $g : \Omega \setminus S_2$ are equivalent if $f(z) = g(z)$ on $\Omega \setminus (S_1 \bigcup S_2)$.
- For $f, g \in \mathcal{M}(\Omega)$, define $f + g$ to be the equivalence class of $(f + g) : \Omega \setminus (S_1 \bigcup S_2)$
- Similarly, fg is the equivalence class of $fg : \Omega \setminus (S_1 \bigcup S_2)$.

The space of all meromorphic functions is a field.

Order of meromorphic functions

The order of a meromorphic function is defined as follows,

- If $z_0 \in S$ is a removable singularity then the order of f at z_0 is the order of the zero at z_0 of f, i.e., $f(z) = (z - z_0)^m g(z)$ then m is the order.
- If $z_0 \in S$ is a pole and the pole is of order m then order of f at z_0 is $-m$.
- If $f \equiv 0$ then $\mathrm{Ord}_{z_0} = \infty$.
- $\mathrm{Ord}_{z_0}(f + g) \geq \min(\mathrm{Ord}_{z_0}(f), \mathrm{Ord}_{z_0}(g))$
- $\mathrm{Ord}_{z_0}(fg) = \mathrm{Ord}_{z_0}(f) + \mathrm{Ord}_{z_0}(g)$

Residue of a function

Residue of a function: Let $f : \Omega \setminus S \to \mathbb{C}$ be a holomorphic function, where Ω is an open set and S is a discrete subset of Ω. Then for $z_0 \in S$, let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$ and $B(z_0, r) = \{z_0\}$. Then in $B(z_0, r) \setminus \{z_0\}$, consider the Laurent series expansion of f given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. We define the residue of f at z_0 to be $\mathrm{Res}(f, z_0) = a_{-1}$.

1. If z_0 is a removable singularity then $\mathrm{Res}(z_0) = 0$.
2. If z_0 is a pole of order m then $(z - z_0)^m f(z) = g(z)$, where $g(z) \neq 0$ on $B(z_0, r) \setminus \{z_0\}$then, $\mathrm{Res}(z_0) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}$.

Residue theorem

Let Ω be an open connected subset of $\mathbb{C}$ and S be a finite subset of Ω and let $f : \Omega \setminus S \to \mathbb{C}$ be a holomorphic function. Let γ be a null homotopic closed curve on Ω. Then,

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{j=1}^{k} W_\gamma(z_j) \mathrm{Res}(f, z_j)$$

where $S = \{z_1, \cdots, z_k\}$ and W_γ is the winding number.

Log derivative

For a holomorphic function $f : \Omega \to \mathbb{C}$. Define the log derivative of f to be the meromorphic function $\frac{f'(z)}{f(z)}$.

1. $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$
2. $\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}$
3. When f has a pole of order m at z_0 then for $f(z) = \frac{g(z)}{(z-z_0)^m}$ the log derivative of f is $\frac{g'(z)}{g(z)} - \frac{m}{(z-z_0)}$

Argument principle

Let $f : \Omega \setminus S \to \mathbb{C}$ be a meromorphic function s.t. f has zeros of order $d_1, \ldots, d_n$ at $z_1, \ldots z_n$ after removing the removable singularities. And f has poles of order $e_1, \ldots, e_m$ at points $w_1, \ldots, w_m$. Let γ be a closed curve which is null homotopic in Ω s.t. the zeros and poles don't lie in the image of γ. Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{i=0}^{n} d_i W_\gamma(z_i) - \sum_{j=1}^{m} e_j W_\gamma(w_j)$$

Rouche's theorem

Let γ be a closed curve which is null homotopic in Ω. Let f, g be functions holomorphic in Ω and $|g(z)| < |f(z)|$ on γ then f and $f + g$ have the same number of zeros counting multiplicities on the interior of $H([0,1] \times [a,b])$ where H is the null homotopy from γ to a constant path.

Branch of the complex logarithm

Let Ω be an open connected subset of $\mathbb{C} \setminus \{0\}$. Define a branch of the logarithm on Ω as a function $f : \Omega \to \mathbb{C}$ s.t. $\exp(f(z)) = z, \forall z \in \Omega$.
For $\Omega = \mathbb{C} \setminus \{\Re(x) \leq 0\}$ define the standard branch to be

$$\mathrm{Log}(z) = \ln |z| + i\mathrm{Arg}(z)$$

As defined above $\mathrm{Log}(z)$ is holomorphic on Ω.

Schwarz lemma

Let $\mathbb{D}$ denote the open unit disc. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function s.t. $f(0) = 0$. Then,

$$|f(z)| \le |z|, \forall z \in \mathbb{D}, \text{ and } |f'(z)| \le 1$$

Also, if $|f(z)| = |z|$ for some $z \in \mathbb{D}$ or if $|f'(0)| = 1$ then $\exists \lambda \in \mathbb{C}, |\lambda| = 1$ s.t. $f(z) = \lambda z$.

Automorphism

A function $f : \Omega \to \Omega$ is an automorphism if f is holomorphic and has a holomorphic inverse.

Automorphisms of the unit disc

Define a function $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$ defined as $\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$.
Let $f : \mathbb{D} \to \mathbb{D}$ be an automorphism. Then there exists $\alpha \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ s.t.

$$f(z) = \lambda \varphi_\alpha(z)$$

Phragmén–Lindelöf method

Let $\Omega = \{z \in \Omega : a < \Re(z) < b\}$. Let $f : \overline{\Omega} \to \mathbb{C}$, s.t. f is continuous on $\overline{\Omega}$ and holomorphic on Ω. Suppose for some $z = x + iy$, we have $|f(z)| < B$ and let $M(x) = \sup\{|f(x+iy)| : -\infty < y < \infty\}$. Then,

$$M(x)^{b-a} \le M(a)^{b-x} M(b)^{x-a}$$

And further

$$|f(z)| \le M(x) \le \max\{M(a), M(b)\} = \sup_{z \in \partial\Omega} |f(z)|$$

Schwarz-Pick theorem

First define $\rho(z, w) = \left|\frac{z-w}{1-\overline{w}z}\right|$ for $z, w \in \mathbb{D}$. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then,

$$\rho(f(z), f(w)) \le \rho(z, w)\ \forall z, w \in \mathbb{D}$$

and,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}\ \forall z \in \mathbb{D}$$

Lifting of maps

Let X, Y, Z be open subsets of $\mathbb{C}$ and let $f : Y \to X$ and $g : Z \to X$ be continuous maps. Then we say, a map $\widetilde{g} : Z \to Y$ is a lift of g w.r.t. f if $f \circ \widetilde{g} = g$.

Uniqueness of lifts: Let X, Y, Z be open connected subsets of $\mathbb{C}$ and let $f : Y \to X$ be a *local* homeomorphism. Let $g : Z \to X$ be a continuous map. Let $\widetilde{g_1}$ and $\widetilde{g_2}$ be lifts of g w.r.t. f and suppose they are equal at some point in Z. Then $\widetilde{g_1} \equiv \widetilde{g_2}$.

- Let $f : Y \to X$ be a holomorphic map s.t. $f'(y) \neq 0$ on Y. Let $g : Z \to X$ be a holomorphic map s.t. $\widetilde{g} : Z \to Y$ is a lift of g w.r.t. f. Then $\widetilde{g}$ is holomorphic.
- Let X, Y be open subsets of $\mathbb{C}$ let, $f : Y \to X$ be a local homeomorphism. Let γ_0, γ_1 be curves in X from z_1 to z_2 which are homotopic. Suppose that for every $s \in [0, 1]$, we can lift $\gamma_s(t) = H(s, t)$ to a path $\widetilde{\gamma_s} : [a, b] \to Y$ w.r.t. f s.t. $\widetilde{\gamma_s}(a) = \widetilde{z_1}$, $\forall s \in [0, 1]$. Then $\widetilde{\gamma_0}, \widetilde{\gamma_1}$ are homotopic in Y.

Covering spaces

Let X, Y be open subsets of $\mathbb{C}$. We say that a continuous map $f : Y \to X$ is a covering map if given $x \in X$ there exists a neighbourhood U of X and open sets $\{V_\alpha\}_{\alpha \in A}$ in Y s.t. $f^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$ (disjoint union of V_α) and $f|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Then Y is called a cover of X.

- Let $f : Y \to X$ be a covering map and $\gamma[a, b] \to X$ be a curve from x_0 to x_1 in X. Suppose $y_0 \in f^{-1}(\{x_0\})$. Then there exists a unique lift $\widetilde{\gamma}[a, b] \to Y$ of γ w.r.t. f s.t. $\widetilde{\gamma}(a) = y_0$.
- For connected X let $f : Y \to X$ be a covering map. Suppose $x_0, x_1 \in X$. Then the cardinality of $f^{-1}(x_0)$ is the same as the cardinality of $f^{-1}(x)$.
- For open subsets X, Y of $\mathbb{C}$ let, $f : Y \to X$ be a covering map from Y to X. Let Z be an open connected subset of $\mathbb{C}$, which is simply connected and locally connected. Suppose $g : Z \to X$ is a continuous map. Then given $z_0 \in C$ and $y_0 \in Y$ s.t. $g(z_0) = f(y_0)$, then there exists a unique lift $\widetilde{g} : Z \to Y$ of g w.r.t f.
- Let Ω be a simply connected, locally connected, open connected subset of $\mathbb{C}$ and $g : \Omega \to \mathbb{C}^*$ be a holomorphic map. Then there exists a lift $\widetilde{g} : \Omega \to \mathbb{C}$ s.t. $\exp(\widetilde{g}) = g$.

Bloch's theorem

- For $f : \mathbb{D} \to \mathbb{C}$ s.t. $f(0) = 0, f'(0) = 1$ and $|f(z)| \le M\ \forall z \in \mathbb{D}$. Then $B(0, \frac{1}{6M}) \subseteq f(\mathbb{D})$.
- Let $f : B(0, R) \to \mathbb{C}$ be holomorphic s.t. $f(0) = 0$, $f'(0) = \mu$ for some $\mu > 0$ and $f|(z)| \le M\ \forall z \in B(0, R)$. Then, $B(0, \frac{R^2\mu^2}{6M}) \subseteq f(B(0, R))$.

Bloch's theorem: Let Ω be an open connected subset of $\mathbb{C}$ s.t. $\overline{\mathbb{D}} \subset \Omega$. Let $f : \Omega \to \mathbb{C}$ s.t. $f(0) = 0$, $f'(0) = 1$. Then there exists a ball B' contained in $\mathbb{D}$ s.t. $f|_{B'}$ is injective and $B(0, \frac{1}{72}) \subseteq f(B') \subseteq f(\mathbb{D})$.

Little Picard's theorem

- Let Ω be an open connected subset of $\mathbb{C}$ which is simply connected. Let $f : \Omega \to \mathbb{C}$ which omits 0 and 1. Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$ s.t. $f(z) = -\exp(\pi i \cosh(2g(z)))$
- The function g as defined above doesn't contain any disk of radius 1.

Little Picard's theorem: If f is an entire function which omits two points, then f is a constant function.