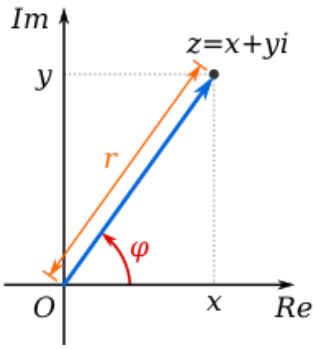
Harold's Complex Variables
Cheat Sheet

20 May 2026

Definitions

Name	Definition or Formula
Imaginary Number	$i = \sqrt{-1}$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$ i is used by mathematicians. j is used by electrical engineers.
Complex Number	 Rectangular Form (x, y): $z = x + iy$ $z = (x, y)$ where $x = \mathrm{Re}\ z;\ y = \mathrm{Im}\ z$ Polar Form (r, θ): $z = r(\cos\theta + i\sin\theta)$ Exponential Form (e^x): $\boldsymbol{z = re^{i\theta}}$ Parametric Form (ρ, θ): $z = z_0 + \rho e^{i\theta}$ $(0 \le \theta \le 2\pi)$ Shorthand: $e^z = \exp(z) = e^x e^{iy}$
Complex Conjugate	$\bar{z} = x - iy$ $\bar{z} = r(\cos\theta - i\sin\theta)$ $\bar{z} = re^{-i\theta}$
Modulus (Magnitude/Absolute Value)	$\lvert z\rvert = \sqrt{x^2 + y^2}$ $\lvert z\rvert = r$ $\lvert z\rvert^2 = (\mathrm{Re}\ z)^2 + (\mathrm{Im}\ z)^2 = z \cdot \bar{z}$
Argument (Angle)	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ If $(-\pi < \theta \le \pi)$ then principle value

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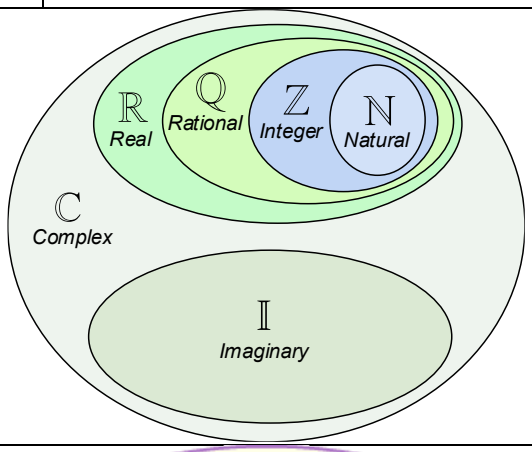
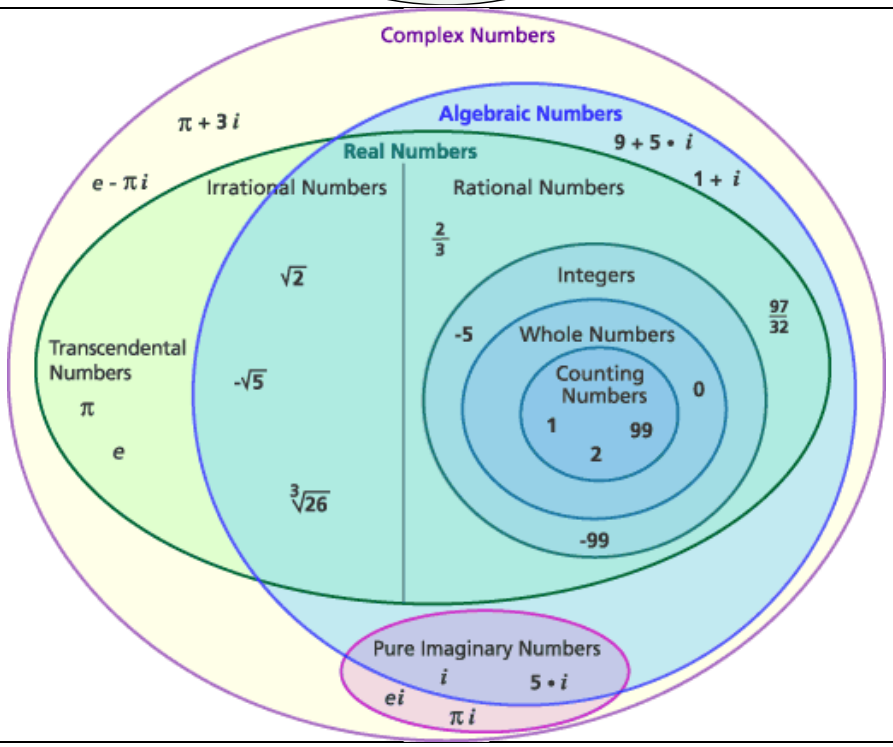
Euler's Formula	$e^{i\theta} = \cos\theta + i\sin\theta$ Examples: $e^{i\frac{\pi}{2}} = i$ $e^{i\pi} = -1$ $e^{-i\frac{\pi}{2}} = -i$ $e^{i2\pi} = 1$
De Moivre's Formula	$z^n = [r\,(\cos\theta + i\sin\theta)]^n$ $\boldsymbol{z^n = r^n(\cos n\theta + i\sin n\theta)}$
Holomorphic Function (Analytic Function)	A complex variable function whose derivative exists at any point.
Meromorphic Function	A complex variable function that is holomorphic except in set points, which are poles.
Entire	A holomorphic function that is holomorphic $\forall z \in C$.
Reflection Principle	$\overline{f(z)} = f(\bar{z})$ If the lower half is the reflection of the upper half over the x-axis.





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Algebraic Properties

Property	Formula
Complex Numbers	$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ $z_1 - z_2 = (x_1 + x_2) - i(y_1 + y_2)$ $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ $\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}, z_2 \neq 0$ $\lvert z_1 \cdot z_2 \cdot z_3 \cdot \ldots \cdot z_n \rvert = \lvert z_1 \rvert \cdot \lvert z_2 \rvert \cdot \lvert z_3 \rvert \cdot \ldots \cdot \lvert z_n \rvert$ $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
Additive Inverses	$-z = (-x, -y)$ $-z = re^{i(\theta+\pi)}$
Multiplicative Inverses	$z^{-1} = \left(\frac{x}{x^2 + y^2}, i\frac{-y}{x^2 + y^2}\right), z \neq 0$ $z^{-1} = \frac{1}{r}e^{-i\theta}$
Complex Conjugates	$\lvert \bar{z} \rvert = \lvert z \rvert$ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$
Triangle Inequality	$\lvert z_1 \pm z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$ $\lvert z_1 \pm z_2 \rvert \geq \lvert z_1 \rvert - \lvert z_2 \rvert$ $\lvert z_1 + z_2 \rvert \geq \big\lvert \lvert z_1 \rvert - \lvert z_2 \rvert \big\rvert$
Exponentials	$z^n = r^n e^{in\theta}$ $z^c = e^{c \log z} = \exp(c \log z)$
Roots	$\boldsymbol{\sqrt[n]{z} = \sqrt[n]{r}\exp\left[i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]}$
Arguments (Angles)	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ $\arg(z_2^{-1}) = -\arg(z_2)$

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Transcendental Properties

Property	Formula
Power	$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$ $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$ $e^z = e^{z + 2\pi k i}$
Logarithms	$\log z = \ln\lvert z\rvert + i \arg z$ $\log e^z = z + 2n\pi i$ $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ $\ln\lvert z_1 z_2\rvert = \ln\lvert z_1\rvert + \ln\lvert z_2\rvert$ $\ln\lvert z_1 z_2\rvert + i \arg(z_1 z_2)$ $= \left(\ln\lvert z_1\rvert + i \arg(z_1)\right) + \left(\ln\lvert z_2\rvert + i \arg(z_2)\right)$
Trigonometric	$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \frac{e^{iz} + e^{-iz}}{2i}$ $\sin^2 z + \cos^2 z = 1$ $\tan^2 z + 1 = \sec^2 z$ $1 + \cot^2 z = \csc^2 z$ $\sin(2z) = 2 \sin z \cos z$ $\cos(2z) = \cos^2 z - \sin^2 z$ $\cos(2z) = 2 \cos^2 z - 1$ $\cos(2z) = 1 - 2 \sin^2 z$
Hyperbolic	$\sinh z = \frac{e^z - e^{-z}}{2}$ $\cosh z = \frac{e^z + e^{-z}}{2}$ $\sin(ix) = i \sinh x$ $\cos(ix) = \cosh x$ $\sin z = \sin x \cosh y + i \cos x \sinh y$ $\cos z = \cos x \cosh y - i \sin x \sinh y$ $\sinh z = \sinh x \cos y + i \cosh x \sin y$ $\cosh z = \cosh x \cos y - i \sinh x \sin y$

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Inverse Trigonometric	$\sin^{-1} z = -i \ln\left[iz + \sqrt{1 - z^2}\right]$ $\cos^{-1} z = -i \ln\left[z + \sqrt{z^2 - 1}\right]$ $\tan^{-1} z = \frac{i}{2} \ln\left[\frac{i + z}{i - z}\right]$ $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1 - z^2}}$ $\frac{d}{dz}\tan^{-1} z = \frac{1}{1 + z^2}$
Inverse Hyperbolic	$\sinh^{-1} z = \ln\left[z + \sqrt{1 + z^2}\right]$ $\cosh^{-1} z = \ln\left[z + \sqrt{z^2 - 1}\right]$ $\tanh^{-1} z = \frac{1}{2} \ln\left[\frac{1 + z}{1 - z}\right]$

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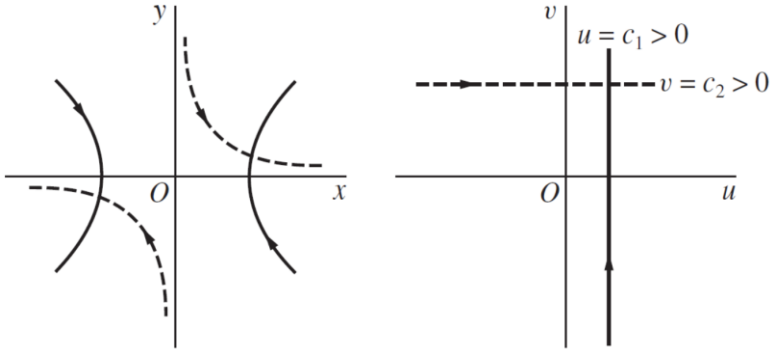
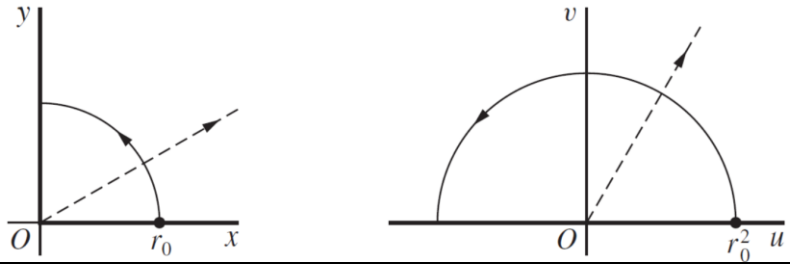
Functions

Name	Formula
Functions	$f(z) = f(x+iy) = u(x,y) + iv(x,y) = u + iv$ $f(z) = f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta) = u + iv$
Conic Mappings	Hyperbola (Rectangular Form): $w = z^2$ $u = x^2 + y^2 = c_1$ $v = 2xy = -2y\sqrt{y^2 + c_1}$  Circle (Polar Form): $w = z^2$ $w = r^2 e^{i2\theta}$ $\rho = r^2$ $\varphi = 2\theta$ 

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Differentiation

Name	Formula
Cauchy-Riemann Equations	Determines whether the given complex-valued function $f(z) = u + iv$ is analytic and differentiable. Rectangular Form: $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at point $z_0 = x_0 + iy_0$ $u_x = v_y, \quad u_y = -v_x$ $f'(z_0) = u_x + iv_x$ where $u_x = \frac{\partial u}{\partial x}$ Polar Form: $f(z) = u(r,\theta) + iv(r,\theta)$ $ru_r = v_\theta, \quad u_\theta = -rv_r$ $f'(z_0) = e^{-i\theta}(u_r + iv_r)$
Laplace's Equation (Harmonic)	$H_{xx}(x,y) + H_{yy}(x,y) = 0$

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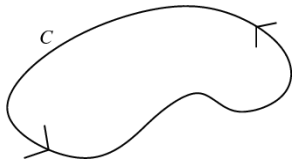
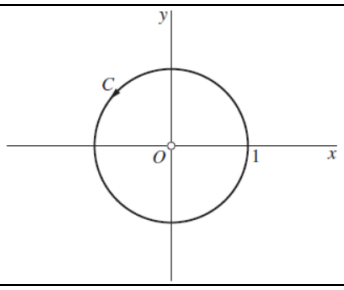
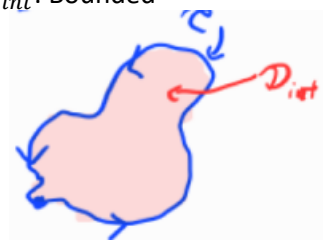
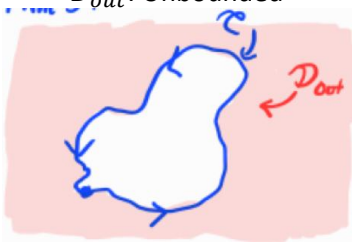
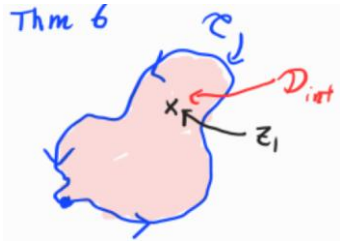
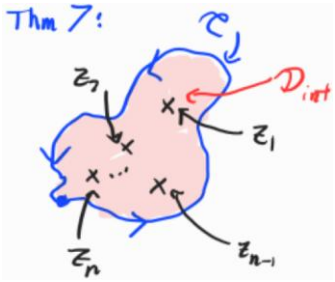
Contours

<table>
<tr><th>Name</th><th colspan="2">Definition</th></tr>
<tr><td>Simple Arc (C)
(Jordan arc)</td><td colspan="2">If arc C does not cross itself; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. E.g., open.</td></tr>
<tr><td>Contour (C)</td><td colspan="2">A closed path in the complex plane.
A piecewise smooth arc consisting of a finite number of smooth arcs joined end to end.</td></tr>
<tr><td rowspan="2">Simple Curve (C)</td><td>A simple arc where $z(b) = z(a)$. E.g., closed.</td><td>Simple closed curve
  </td></tr>
<tr><td>Simple closed curve C defaults to a circle, $|z - 0| = r$, centered at 0 with radius r and interval $[0, 2\pi]$, oriented counterclockwise.

$C(r) = \{z \in \mathbb{C} : |z| = r\}$</td><td>  </td></tr>
<tr><td>Positively Oriented</td><td colspan="2">a simple closed curve, or a Jordan curve, is positively oriented when it is in the counterclockwise direction.</td></tr>
<tr><td>Simply Connected Domain</td><td colspan="2">D is a domain such that every simple closed contour within it encloses only points of D.</td></tr>
<tr><td>Branch Cut</td><td colspan="2">A portion of a line or curve that is introduced to define a branch F of a multiple-valued function f.</td></tr>
<tr><td>Regions Bound by Curve C</td><td>D_{int}: Bounded
  </td><td>D_{out}: Unbounded
  </td></tr>
<tr><td>Closed, Simple, Counter-Clockwise Oriented Curve</td><td>One Point, Simple Pole:
  </td><td>Multiple Points, Simple Poles:
  </td></tr>
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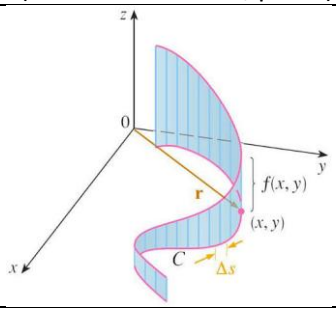
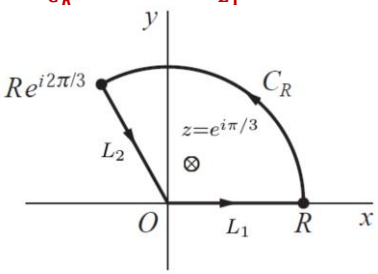
Integration

Name	Formula
Complex Integration	$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C udx - vdy + i\int_C vdx - udy$ "*No corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.*" (Brown & Churchill, p.125)
Contour Integral (Complex $\mathbb{C}$)	Called a Line Integral if Real $\mathbb{R}$ numbers. $\boldsymbol{\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt}$ Change of Contour direction: $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$ 
Fundamental Theorem of Calculus with Contour Integral	$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)\,z'(t)dt = F\big(z(t)\big)\big]_a^b$ $= F(z(b)) - F(z(a)) = F(z_2) - F(z_1)$ Since $z(b) = z_2$ and $z(a) = z_1$. where z_0 is a point within the contour C.
Morera's Theorem	If $\int_C f(z)\,dz = 0$ then, f is **Holomorphic** over $\mathbb{R}$.
Cyclic Integral	$\oint_C f(z)\,dz$ Integral over a **closed** contour, meaning the curve returns to its initial position ($a = b$). For circular contours, $\oint_{C_R} f(z)\,dz = \int_0^{2\pi} f(z)\,dz$ For non-circular contours, $\boldsymbol{\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz}$  Parameterize the arcs and identify the bounds of integration. $L_1: z = re^{i0}, \quad 0 \le r \le R$ $L_2: z = re^{i2\pi/3}, \quad R \le r \le 0$ $C_R: z = Re^{i\theta}, \quad 0 \le \theta \le 2\pi/3$

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Cauchy-Goursat Theorem (Cauchy's Integral Theorem)	D_{int}: If C is closed, i.e., $z_0 = z_1$, then $$\oint_C f(z)\,dz = 0$$ D_{out}: Outside of closed C, at infinity (∞): If $$\lim_{z\to\infty} f(z)\,z = 0$$ Then $$\oint_C f(z)\,dz = 0$$
Cauchy Integral Formula	Turns a contour integral into a derivative. Simple: $$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$ General: $$\boldsymbol{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz}$$ where $n = 0, 1, 2, \ldots;\ 0! = 1;\ f^{(0)}(z_0) = f(z_0)$.
Jordan's Lemma	Estimation Lemma: $$\left\lvert\int_C f(z)\,dz\right\rvert \le \text{length}(C)\cdot\max_{z\in C}\lvert f(z)\rvert$$ Common Application: If $$\lim_{z\to\infty} f(z)\,z = 0$$ Then $$\int_{-\infty}^{\infty} f(at+b)\,a\,dt\,x = \int_L f(z)\,dz = \lim_{R\to\infty}\oint_{C_R} f(z)\,dz$$ where line $L = \{at + b\}, -\infty < t < \infty$ and R = semi-circle radius along this line.

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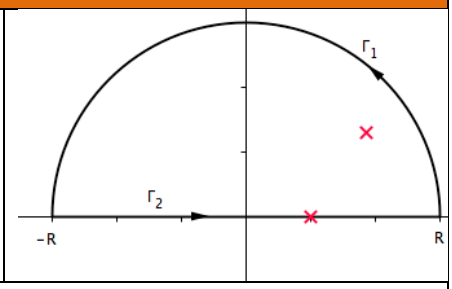
Poles and Residues

Name	Formula
Poles	Roots in the denominator of a complex function that is holomorphic (complex differential). E.g., Singularity, vertical asymptote. 
	Poles are zeros in the denominator of $f(z)$. $$f(z) = \frac{\varphi(z)}{(z - z_0)^m}$$ Simple Pole: $$z_0 \text{ is a pole of order } 1 \ (m = 1).$$ High-Order Pole: $$z_0 \text{ is a pole of order } m \ (m = 2, 3, 4, \dots).$$ **Theorem**: If z_0 is a pole of function f, then $$\lim_{z \to z_0} f(z) = \infty.$$
Residue	$$\mathbf{Res}_{z_1}(f) = \lim_{z \to z_1} (z - z_1)\, f(z)$$ **Observation**: Since z_1 is a simple pole, then the $\text{Res}_{z_1}(f)$ turns the function f into a function with a hole or hollow point. The limit makes the remaining function appear continuous. "*Remove the pole and cover the hole.*" **Tip:** If Laurent Series centered at 0, then $Res_{z_1}(f) = a$ the z^{-1} term of $f(z) = \cdots + az^{-1} + \cdots$.
Cauchy's Residue Theorem (Simple Poles)	One Point, Simple Pole inside Contour C: If exists $$\text{Res}_{z_1}(f) = \lim_{z \to z_1} (z - z_1)\, f(z)$$ Then $$\oint_C f(z)\, dz = 2\pi i \, \text{Res}_{z_1} f(z)$$

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	Multiple Points, Simple Poles inside Contour C: If these exist $$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) =$$ $$Res_{z_1}(f) = \lim_{z \to z_1} (z - z_1)\, f(z)$$ + ... + $$Res_{z_n}(f) = \lim_{z \to z_n} (z - z_n)\, f(z)$$ Then $$\boldsymbol{\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)}$$
	Special Case: If $f(z)$ is even, then $$\int_{-\infty}^{\infty} f(z)\, dz = \frac{1}{2} \int_{0}^{\infty} f(z)\, dz$$
Residue of High-Order Poles	General: Works with higher-order poles. $$\text{If } f(z) = \frac{\varphi(z)}{(z - z_0)^m} \quad (m = 1, 2, 3, \dots)$$ where $\varphi(z)$ is analytic and nonzero at z_0, then $$\boldsymbol{\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \quad (m = 1, 2, \dots)}$$ Simple: $$\operatorname*{Res}_{z=z_0} f(z) = \varphi(z_0) \text{ when } m = 1$$ since $\varphi^{(0)}(z_0) = \varphi(z_0)$ and $0! = 1$. Tip: $$\text{If } f(z) = \frac{g(z)}{(z - z_0)^m (z - z_1)^p \cdots (z - z_{n-1})^q}$$ $$\text{then choose } \varphi(z) = \frac{g(z)}{(z - z_1)^p \cdots (z - z_{n-1})^q}.$$
Residue of Simple Poles (shortcut)	**Theorem**: Let two functions p and q be analytic at a point z_0. If $$p(z_0) \neq 0,$$ $$q(z_0) = 0, \text{and}$$ $$q'(z_0) \neq 0,$$ then z_0 is a simple **pole** of the quotient $p(z)/q(z)$ and $$\boldsymbol{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}.$$
Zeros and Poles	**Theorem**: Suppose that: (a) two functions p and q are analytic at a point z_0; (b) $p(z_0) \neq 0$ and q has a **zero** of order m at z_0. Then the quotient $p(z)/q(z)$ has a **pole** of order m at z_0.

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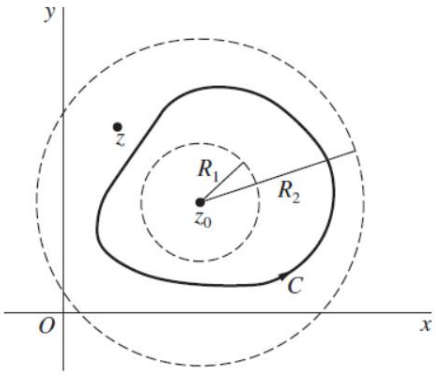
Series

Name	Formula
Liouville's Theorem	If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.
Fundamental Theorem of Algebra	$P(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0, \qquad (a_n \neq 0, n \geq 1)$ $P(z) = c(z - z_n) \ldots (z - z_2)(z - z_1)$ Any polynomial of degree n has at least one zero in the complex plane. That is, there exists at least one point z_0 such that $P(z_0) = 0$.
Maximum Modulus Principle	**Theorem:** If a function f is analytic and not constant in a given domain D, then the modulus $\lvert f(z)\rvert$ has no maximum value in D. That is, there is no point z_0 in the domain such that $\lvert f(z)\rvert \leq \lvert f(z_0)\rvert$ for all points z in it.
	Corollary: Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $\lvert f(z)\rvert$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.
Complex Variable Convergence	$\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ where $z_n = x_n + iy_n$
Complex Series Convergence	$\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$ where $S = X + iY$
Series Convergence	**Corollary 1**: If a series of complex numbers converges, the n[th] term converges to zero as n tends to infinity. **Corollary 2:** The absolute convergence of a series of complex numbers implies the convergence of that series.
Annular Domain $R_1 < \lvert z - z_0\rvert < R_2$	
Transcendental Series	See Harold's Taylor Series Cheat Sheet for a comprehensive list of the Maclaurin series of all transcendental functions.

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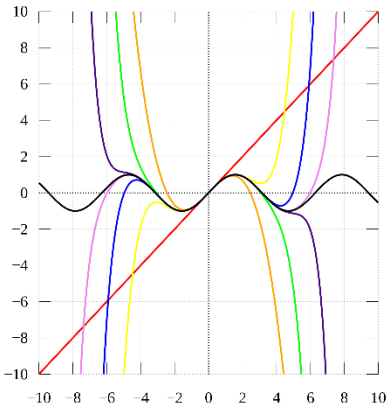
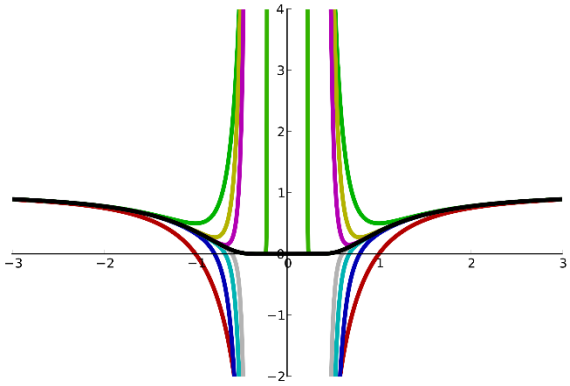
Taylor Series	$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n \quad \text{where disk } (|z - z_0| < R_0)$$ $$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{where } (n = 0, 1, 2, \dots)$$ The series converges to $f(z)$ when z lies in the stated open disk. If $z_0 = 0$, then **Maclaurin series**. 
Laurent Series	$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{\boldsymbol{n=1}}^{\infty} \frac{\boldsymbol{b_n}}{\boldsymbol{(z - z_0)^n}}$$ $$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \text{ where } (n = 0, 1, 2, \dots)$$ $$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{-n+1}}\,dz \quad \text{where } (n = 1, 2, 3, \dots)$$ $$\text{where } (R_1 < |z - z_0| < R_2)$$  Taylor Series Form: $$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$ $$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz = \frac{f^{(n)}(z_0)}{n!} \quad \text{where } (n = 0, 1, 2, \dots)$$ $$\text{and } (R_1 < |z - z_0| < R_2)$$ If no poles, then **Taylor series**.

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Power Series

Name	Formula
Power Series	$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $(\lvert z - z_0 \rvert < R)$
Absolute and Uniform Convergence	**Theorem 1**: If a power series converges when $z = z_1 \ (z_1 \neq z_0)$, then it is **absolutely** convergent at each point z in the open disk $\lvert z - z_0 \rvert < R_1$ where $R_1 = \lvert z_1 - z_0 \rvert$.
	Theorem 2: If z_1 is a point inside the circle of convergence $\lvert z - z_0 \rvert = R$ of a power series then that series must be **uniformly** convergent in the closed disk $\lvert z - z_0 \rvert \leq R_1$ where $R_1 = \lvert z_1 - z_0 \rvert$.
Continuity of Sums	**Theorem**: A power series represents a **continuous** function $S(z)$ at each point inside its circle of convergence $\lvert z - z_0 \rvert = R$.
Integration	**Theorem**: Let C denote any contour interior to the circle of convergence of the power series and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be **integrated** term by term over C; that is, $\int_C g(z)\, S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)\, (z - z_0)^n dz$.
Differentiation	**Theorem**: The power series can be **differentiated** term by term. That is, at each point z interior to the circle of convergence of that series, $S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$.
Leibniz's Rule for the nth Derivative	$[f(z)\, g(z)]^{(n)} = \sum_{n=0}^{\infty} \binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$ where $(n = 1, 2, \dots)$ $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ where $(k = 0, 1, 2, \dots, n)$
Uniqueness Representations	**Theorem 1**: If a power series converges to $f(z)$ at all points interior to some circle $\lvert z - z_0 \rvert = R$, then it is the **Taylor series** expansion for f in powers of $z - z_0$.
	Theorem 2: If a series $\sum_{\boldsymbol{n=-\infty}}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the **Laurent series** expansion for f in powers of $z - z_0$ for that domain.
Multiplication	Let $f(z), g(z), h(z),$ and $k(z)$ all be different power series.
	$f(z)g(z) = h(z)$
Division	$\frac{f(z)}{g(z)} = k(z)$

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College Course

- **Course**: NYU MATH-UY-4434: Applied Complex Variables, 2024.
- **Textbook**: Complex Variables and Applications, 9th Edition, Chapters 1-7, James Ward Brown & Ruel V. Churchill, McGraw-Hill Education, 2014.

References

- Bhoris Dhanjal, Bhorice2099 on Reddit (2021, 11 October). Imgur, Introductory Complex Analysis Cheat Sheet v2.

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Introductory Complex Analysis Cheat Sheet

Field of Complex Numbers

We construct the field of complex numbers as the following quotient ring, $\mathbb{C} = \mathbb{R}[x]/\langle x^2+1\rangle$

Algebra of Complex Numbers

- Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$
- Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$
- Division: $\dfrac{a+ib}{c+id} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$
- Square root: $\sqrt{a+ib} = \pm\left(\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}} + i\dfrac{b}{|b|}\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}}\right)$
- $\Re(a+ib) = a, \Im(a+ib) = b$

Conjugation, Absolute Value

- **Complex conjugation:** $\overline{a+ib} = a-ib$
  - $\overline{a+b} = \overline{a}+\overline{b}$
  - $\overline{ab} = \overline{a}\cdot\overline{b}$

  Geometrically, conjugation is reflection over the real axis.
- **Absolute value:** $|a| = +\sqrt{a\overline{a}}$
  - $|ab| = |a|\cdot|b|$
  - $|a+b|^2 = |a|^2+|b|^2+2\Re(a\overline{b})$
  - $|a-b|^2 = |a|^2+|b|^2-2\Re(a\overline{b})$
  - $|a+b|^2+|a-b|^2 = 2(|a|^2+|b|^2)$

  The absolute value function forms the metric on $\mathbb{C}$. $\mathbb{C}$ is complete under this metric.

Basic Topological definitions in $\mathbb{C}$

Some basic results:

- For $z_0 \in \mathbb{C}, r > 0$ we denote the ball (i.e. disk) of radius r around z_0 to be $B(z_0, r) = \{z \in \mathbb{C} \mid |z-z_0| < r\}$
- A point $z \in \mathbb{C}$ is a **limit point** of $E \subseteq \mathbb{C}$ if $\forall \varepsilon > 0$, $B(z,\varepsilon) \cap E$ contains a point other than z.
- A subset $E \subseteq \mathbb{C}$ is said to be **open** if $\forall z \in E, \exists\, r > 0$, s.t. $B(z,r) \subset E$.
- A subset $E \subseteq \mathbb{C}$ is said to be **closed**, if $\mathbb{C}\setminus E$ is open in C. Or equivalently a set which contains all its limit points.

Some properties of open sets:

- $\mathbb{C}$ and $\emptyset$ are open subsets of $\mathbb{C}$.
- All finite intersections of open sets are open sets.
- The collection of all open sets on $\mathbb{C}$ form a topology on $\mathbb{C}$.

Interior, closure, density

- **Interior:** Let $E \subseteq \mathbb{C}$. The interior of E is defined as, E°=set of all interior points of E, or equivalently, $\cup\{\Omega \mid \Omega \subseteq E \wedge \Omega \text{ is open in } \mathbb{C}\}$
- **Closure:** Let $E \subseteq \mathbb{C}$. The closure of E is defined as $\hat{\{}F \mid E \subseteq F \wedge F \text{ is closed in } \mathbb{C}\}$
- **Density:** Let $E \subseteq D$, the closure of E in D is D. Then E is called dense in D.

Path : A path in a metric space from a point $x \in X$ to $y \in Y$ is a continuous mapping $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$.

Separated and Connected

For a metric space (X, d).

- **Separated:** X is separated if $\exists$ disjoint non-empty open subsets A, B of X s.t. $X = A \cup B$.
- **Connected:**
  - X is connected if it has no separation.
  - X is connected $\iff$ X does not contain a proper subset of X which is both open and closed in X.
  - Continuous functions preserve connectedness.
  - An open subset $\Omega \in \mathbb{C}$ is connected $\iff$ for $z, w \in \Omega$, there exists a path from z to w.

Basic Topological definitions in $\mathbb{C}$ contd.

Open cover: Let (X, d) be a metric space and E be a collection of open sets in X. We say that $\mathscr{U}$ is an open cover of a subset $K \subseteq X$, if $K \subset \bigcup\{\mathscr{U} \mid \mathscr{U} \in E\}$

Compactness: For some $K \subseteq X$ is compact if for every open cover E of K, there exists $E_1, \cdots . E_n \in E$ s.t. $K \subset U_{i=1}^n E_n$, i.e. it is compact if it has a finite open cover.

- In a metric space, a compact set is closed.
- A closed subset of a compact set is closed.

Limit point compact: We say a metric space X is limit point compact if every infinite subset of X has a limit point.

- If X is a compact metric space, then it is also limit point compact.

Sequentially compact: We say a metric space X is sequentially compact if every sequence has a convergent sub-sequence.

- If X is limit point compact then X is sequentially compact.
- Let X be sequentially compact, then X is a compact metric space.

Lebesgue number lemma: Let X be sequentially compact, and let $\mathscr{U}$ be an open cover of X. Then $\exists\, \delta > 0$ s.t. for $x \in X$, $\exists\, u \in \mathscr{U}$ s.t. $B(x,\delta) \subseteq u$.

Isometries on the Complex Plane

A function $f : \mathbb{C} \to \mathbb{C}$ is called an **isometry** if $|f(z)-f(w)| = |z-w|\,, \forall z, w \in \mathbb{C}$.

- Let f be an isometry s.t. $f(0) = 0$, then the inner product $\langle f(z), f(w)\rangle = \langle z, w\rangle\,, \forall z, w \in \mathbb{C}$.
- If f is an isometry s.t. $f(0) = 0$ then f is a linear map.
- The standard argument for $a+ib \in \mathbb{C}$, $\text{Arg}(a+ib) = \tan^{-1}\frac{b}{a}$

Functions on the Complex Plane

Uniform convergence: Let $\Omega \subseteq \mathbb{C}$ and $f_1, \cdots, f_n : \Omega \to \mathbb{C}$ be a set of functions on Ω. We say, $\{f\}_{n\in\mathbb{N}}$ converges uniformly to f if given $\varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon, \forall x \in \Omega$ and $n \geq N$.

Complex exponential: For $z \in C$, $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Trigonometric functions: For $z \in \mathbb{C}$, $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(x) = \frac{e^{ix}-e^{ix}}{2}$

Hyperbolic trigonometric functions: For $z \in \mathbb{C}$, $\cosh(x) = \frac{e^x+e^{-x}}{2}$ and $\sinh(z) = \frac{e^x-e^{-x}}{2}$

Complex differentiability

Complex derivative: Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$, we say that f is complex differentiable at a point $z_0 \in \Omega$ if z_0 is an interior point and the following limit exists $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. The limit is denoted as $f'(z_0)$ or $\frac{\mathrm{d}f(z)}{\mathrm{d}z}$.

Holomorphic functions: If $f : \Omega \to \mathbb{C}$ is complex differentiable at every point $z \in \Omega$, then f is said to be a holomorphic on Ω. **Entire function:** Functions which are complex differentiable on $\mathbb{C}$ are called entire functions.

- Complex differentiability implies continuity.
- Complex derivatives of a function are linear transformations.
- **Product rule:** If $f, g : \Omega \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$. Then fg is complex differentiable at z_0 with derivative $f'(z_0)g(z_0) + g'(z_0)f(z_0)$.
- **Quotient rule:** If $f, g : \Omega \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, and g doesn't vanish at z_0. Then $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0)-g'(z_0)f(z_0)}{g(z_0)^2}$
- **Chain rule:** If $f : \Omega \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are complex differentiable at $z_0 \in \Omega$, and $f(\Omega) \subseteq D$. Then $g(f(x))'(z_0) = g'(f(z_0))f'(z_0)$

Power Series

Formal Power Series: A formal power series around $z_0 \in \mathbb{C}$ is a formal expansion $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, where $a_n \in \mathbb{C}$ and z is indeterminate.

Radius of convergence: For a formal power series $\sum a_n(z-z_0)^n$ the radius of convergence $R \in [0,\infty]$ given by $R = \liminf_{n\to\infty} |a_n|^{-1/n}$. Using the ratio test is identical i.e. $R = \liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$.

- The series converges absolutely when $z \in B(z_0, R)$, and for $r < R$, the series converges uniformly, else if $|z-z_0| > R$ the series diverges.
- Let $z \in \mathbb{C}$ s.t. $|z-z_0| > R$, then $\exists$ infinitely many $n \in N$ s.t. $|a_n|^{-1/n} < |z-z_0|$.

Abel's Theorem: Let $F(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with a positive radius of convergence R, suppose $z_1 = z_0 + Re^{i\theta}$ be a point s.t. $F(z_1)$ converges. Then $\lim_{r\to R^-} F(z_0 + re^{i\theta}) = F(z_1)$

Differentiation of Power Series

Let $F(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ be a power series around z_0 with a radius of convergence R. Then F is **holomorphic** in $B(z_0, R)$.

- $F(x)' = \sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1}$ with same radius of convergence R.
- $a_n = \frac{F^n(z_0)}{n!}$

Cauchy product of two power series: For power series $F(z) = \sum a_n(z-z_0)^n$ and $G(z) = \sum a_n(z-z_0)^n$ with degree of convergence at least R. Then the Cauchy product $F(z)G(z) = \sum c_n(z-z_0)^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ also has degree of convergence at least R.

Cauchy-Riemann Differential Equations

For a complex function $f(z) = u(z) + iv(z)$,
$f'(x) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ or $f'(x) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Therefore, we get the two **Cauchy-Riemann Differential equations**,

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$
- $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

A function is holomorphic **iff** it satisfies the Cauchy-Riemann equations.

Wirtinger derivatives:

- $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$
- $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$

If f is holomorphic at z_0 then, $\frac{\partial f}{\partial \overline{z}} = 0$ and $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$

Harmonic Functions

Laplacian: Define $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Harmonic function: Let $u : \Omega \to \mathbb{R}$ be a twice differentiable function. We say that u is a harmonic function if $\Delta u = 0$

For any holomorphic function f, $\Re(f), \Im(f)$ are examples of harmonic functions, but there are harmonic functions which are not holomorphic.

Boundary of a set: For a metric space X, $\Omega \in X$,
the boundary of $\Omega = \partial\Omega = \overline{\Omega} \cap \overline{\Omega^C}$

Maximum principle for harmonic functions: Let $u : \Omega \to \mathbb{R}$ be a twice differentiable harmonic function. Let $k \subset \Omega$ be a compact sub set of Ω. Then, $\sup_{z\in k} u(z) = \sup_{z\in\partial k} u(z)$ and $\inf_{z\in k} u(z) = \inf_{z\in\partial k} u(z)$

Maximum principle for holomorphic functions: Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, for compact $k \subseteq \Omega$, we have, $\sup_{z\in k} |f(z)| = \sup_{\partial k} |f(z)|$

Harmonic conjugate: Let $u : \Omega \to \mathbb{R}$ be a twice differentiable harmonic function. We say that $v : \Omega \to \mathbb{R}$ is a harmonic conjugate of u if $f = u + iv$ is holomorphic.

- For a harmonic function from $\mathbb{C}$ to $\mathbb{R}$ there exists a uniquely determined harmonic conjugate from $\mathbb{C}$ to $\mathbb{R}$ (up to constants).

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Riemann Sphere

Extended complex plane: $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$
Consider S^2, associate every point $z = x + iy$ with a line L that connects to the point $P = (0, 0, 1)$. $L = (1 - t)z + tP$, where $t \in \mathbb{R}$.
The point at which L for some z touches S^2 is given as $\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$, associate P with ∞. This gives a stereographic projection of the complex plane unto S^2. This sphere is known as the Riemann sphere.

Möbius transformations

A map $S(z) = \dfrac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ is called a Möbius transformation if $ad - bc \neq 0$.
Every mobius transformation is holomorphic at $\mathbb{C} \setminus \{-d/c\}$, i.e. every point other than is zero.

- The set of all mobius transformations is a group under transposition.
- S forms a bijection with $\widehat{\mathbb{C}}$

Every mobius transformation can be written as composition of,

1. Translation: $S(z) = z + b, b \in \mathbb{C}$
2. Dilation: $S(z) = az, a \neq 0, a = e^{i\theta}$
3. Inversion: $S(z) = 1/z$

Curves in $\mathbb{C}$

A continuous parametrized curve is a continuous map $\gamma : [a, b] \to \mathbb{C}$ for $a, b \in \mathbb{R}$.

- If $a = b$ the curve is trivial.
- $\gamma(a)$ is initial point and $\gamma(b)$ is terminal point.
- γ is said to be closed if $\gamma(a) = \gamma(b)$.
- γ is said to be simple if it is injective, i.e. doesn't "cross" itself.
- A curve $-\gamma$ is a reversal of γ if $\gamma : [-a, -b] \to \mathbb{C}$ and if $-\gamma(t) = \gamma(-t)$
- γ is said to be continuously differentiable if $\gamma'(t_0)$ (defined usually) exists and is continuous.

Reparametrization: We say a curve $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ is a continuous reparametrization of $\gamma_1 : [a_1, b_1] \to \mathbb{C}$, if there exists a homeomorphism $\varphi : [a_1, b_1] \to [a_2, b_2]$ s.t.$\varphi(a_1) = a_2, \varphi(b_1) = b_2$ and $\gamma_2(\varphi(t)) = \gamma_1(t) \forall t \in [a_1, b_1]$.

- Reparametrization is an equivalence relation.

Arc length: Arc length of curve $\gamma = |\gamma| = \sup \sum_{i=0}^{n} |\gamma(x_{i+1} - \gamma(x_i))|$ for all partitions of $[a, b]$.

- A curve that has a finite arc length is called **rectifiable**.
- $|\gamma| = \displaystyle\int_a^b |\gamma'(t)|\, dt$

First Fundamental Theorem of Calculus

Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $F : \Omega \to \mathbb{C}$ be called the antiderivative of f, i.e. F is holomorphic in Ω and $F'(z) = f(z), \forall z \in \Omega$. For a rectifiable curve γ, $\int_\gamma f(z)dz = F(z_1) - F(z_0)$, where z_0 is the initial point and z_1 is the terminal point.

Second Fundamental Theorem of Calculus

Let $f : \Omega \to \mathbb{C}$ be a continuous function such that $\int_\gamma f = 0$. Whenever γ is a closed polygonal path contained in Ω. For fixed $z_0 \in \Omega$, define a path γ_1 from z_0 to z_1 such that $F(z_1) = \int_{\gamma_1} f(z)\, dz$. Then F is a well defined holomorphic function s.t. $F'(z_1) = f(z_1)\ \forall z_1 \in \Omega$

Properties of complex integration

For continuously differentiable curves $\gamma : [a, b] \to \mathbb{C}$, and $\sigma : [b, c] \to \mathbb{C}$

- For a reparametrization $\widetilde{\gamma}$ of γ we can say that $\int_\gamma f(z)\, dz = \int_{\widetilde{\gamma}} f(z)\, dz$
- $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$
- $\int_{\gamma+\sigma} f(z)\, dz = \int_\gamma f(z)\, dz + \int_\sigma f(z)\, dz$
- $\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$
- If f is bounded by M then $\int_\gamma f(z)\, dz \leq M|\gamma|$
- For $c \in \mathbb{C}$, we have, $\int_\gamma (cf + g)(z)\, dz = c\int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz$

Homotopy of curves

Consider two curves $\gamma_0, \gamma_1 \to \Omega$ with the same initial and end point $[a, b]$.
We say that γ_0 is homotopic to γ_1 ($\gamma_0 \sim \gamma_1$) if there exists a continuous map $H : [0, 1] \times [a, b] \to \Omega$ s.t. $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t),\ \forall t \in [a, b]$.
$H(s, a) = z_0, H(s, b) = z_1\ \forall s \in [0, 1]$
For **closed curves** γ_0 at z_0 and γ_1 at z_1, we say that γ_0 is homotopic to γ_1 as closed curves if there exists a continuous map $H : [0, 1] \times [a, b] \to \Omega$, s.t. $H(0, t) = \gamma_0(t), H(1, t) = \gamma_1(t),\ \forall t \in [a, b]$. And $H(s, a) = H(s, b),\ \forall s \in [0, 1]$.

- Homotopy is an equivalence relation.

Cauchy-Goursat Theorem

Cauchy-Goursat theorem: If a curve γ_0 is homotopic to a reparametrization of γ_1 then, the integral of some function $f : \Omega \to \mathbb{C}$ is homotopy invariant, i.e., $\displaystyle\int_{\gamma_0} f = \int_{\gamma_1} f$
Alternative statement: Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, and $\gamma_0 : [a, b] \to \Omega$ is a rectifiable curve which is null-homotopic (i.e. homotopic to a constant map). Then, $\displaystyle\int_{\gamma_0} f(z)\, dz = 0$

Cauchy's theorem for convex domains

Let $\Omega \subseteq \mathbb{C}$ be a convex and open set and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then f has an anti derivative F on Ω, and if γ is a closed rectifiable curve on Ω then $\int_\gamma f = 0$.

Cauchy's integral formula

Let $f : \Omega \to \mathbb{C}$ be holomorphic. Fix $z_0 \in \Omega$ and let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$. Suppose γ is a closed curve in $\Omega \setminus \{z_0\}$ s.t. γ is homotopic to a reparametrization to γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz$$

Complex analytic function

An alternative statement, we say $f : \Omega \to \mathbb{C}$ is complex analytical if given $z_0 \in \Omega, \exists\, B(z_0, r) \subseteq \Omega$ s.t. the formal power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to f.
Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Suppose for $z_0 \in \Omega, \overline{B(z_0, r)} \subset \Omega$, then for every $n \in \mathbb{N}$, let $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ where $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(z_0, r)$ to $f(z)$.
Corollary: If $f : \Omega \to \mathbb{C}$ is holomorphic then f' is also holomorphic. Therefore f is infinitely differentiable.

Factor theorem for analytic function

For a analytic function $f : \Omega \to \mathbb{C}$ s.t. $f(z_0) = 0$ at $z_0 \in \Omega, \exists$ a unique analytic function $g : \Omega \to \mathbb{C}$ s.t. $f(z) = (z - z_0)g(z)$

Principle of analytical continuation

- Let Ω be open and connected subset of $\mathbb{C}$. and $f, g : \Omega \to \mathbb{C}$ be analytic functions on Ω. Suppose f, g agree on a non-empty subset of Ω, and this subset has an accumulation point. Then $f \equiv g$ on Ω.
- A consequence to this is that, non-trivial holomorphic functions have isolated zeros.

Higher-order Cauchy integral formula

Let $f : \Omega \to \mathbb{C}$ be analytic on Ω and $z_0 \in \Omega$ with $\overline{B(z_0, r)} \subseteq \Omega$. Let γ be a closed curve in $\Omega \setminus \{z_0\}$ that is homotopic to a reparametrization of γ_1 where $\gamma_1(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Cauchy estimates: If $|f(z)| \leq M\ \forall z \in \gamma([0, 2\pi])$ then, $\forall n \in \mathbb{N}$, then we have $|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$

Liouville's Theorem

Let f be a entire function which is bounded. Then f is a constant function.

Fundamental Theorem of Algebra

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a non constant polynomial s.t. $a_i \in \mathbb{C}, a_n \neq 0$. Then $\exists z_1, z_2, \ldots, z_n$ s.t. $p(z) = a_n(z - z_1)\ldots(z - z_n)$.

Morera's Theorem

Let $f : \Omega \to \mathbb{C}$ be a continuous function such that, $\int_\gamma f(z)\, dz = 0, \forall$ closed polygonal paths $\gamma \in \Omega$. Then f is holomorphic on Ω.

Uniform limit of holomorphic functions

Let $f_n : \Omega \to \mathbb{C}$ be a holomorphic on $\Omega, \forall n \in \mathbb{N}$ s.t. f_n converges uniformly on compact sets to f. Then f is holomorphic.

Winding number

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve and let z_0 be a point not in the image of γ. Then the winding number of γ around z_0 is

$$W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}$$

- Winding number is invariant over homotopy.
- Let z_0 be a point not in the image of γ then $\exists r > 0$ s.t. for $z \in B(z_0, r), W_\gamma(z_0) = W_\gamma(z)$
- The winding number is always an integer.
- The winding number is locally constant.

Generalized Cauchy Integral formula: Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and $\gamma : [a, b] \to \Omega$ be a closed curve which is null homotopic. Then for z_0 not in the image of γ,

$$f(z_0)W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)}\, dz$$

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Open Mapping Theorem

- $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then $G : \Omega \times \Omega \to \mathbb{C}$ given by

$$G(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{when } z \neq w \\ f'(z) & \text{when } z = w \end{cases}$$

  then G is continuous.
- Let $f : \Omega \to \mathbb{C}$ be holomorphic on some open set. Suppose $z_0 \in \Omega$ s.t. $f'(z_0) \neq 0$. Then $\exists$ a neighbourhood U of $z_0 \in \Omega$ s.t. f restricted to U is injective. And $V = f(U)$ is an open set and the inverse $g : V \to U$ of f is holomorphic.
- Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on open, connected set Ω. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. Then $\exists$ a neighbourhood U of z_0 and bijective holomorphic function φ on U s.t. $f(z) = w_0 + (\varphi(z))^m$ for $z \in U$ and some integer $m > 0$. And φ maps U unto $B(0,r)$ for some $r > 0$.

Open Mapping Theorem: Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function on open connected set Ω, then $f(\Omega)$ is an open set.

Schwarz reflection principle

Let Ω be a open connected set which is symmetric w.r.t $\mathbb{R}$. Then define the following,

- $\Omega_+ = \{z \in \Omega \mid \Im(z) > 0\}$
- $\Omega_- = \{z \in \Omega \mid \Im(z) < 0\}$
- $I = \{z \in \Omega \mid \Im(z) = 0\}$

Schwarz reflection principle: Let Ω be defined as above. Then if $f : \Omega_+ \bigcup I \to \mathbb{C}$ which is continuous on $\Omega_+ \bigcup I$ and holomorphic on Ω_+. Suppose for $f(x) \in \mathbb{R}$, $\forall x \in I$ then there exists $g : \Omega \to \mathbb{C}$ holomorphic on Ω s.t. $g(z) = f(z)$ for $z \in \Omega_+ \bigcup I$

Singularity of a holomorphic function

- **Isolated singularity:** If f is holomorphic on $B(z_0, R) \setminus \{z_0\}$ for some $R > 0$ then z_0 is called an isolated singularity.
- **Removable singularity:** Let z_0 be an isolated singularity of a holomorphic function f as defined above. It is called removable if there exists holomorphic function g on $B(z_0, R)$ s.t. $g(z) = f(z)$ on $B(z_0, R) \setminus \{z_0\}$.
- **Riemann removable singularity theorem:** Let z_0 be an isolated singularity of a function f, then z_0 is a removable singularity if and only if f is locally bounded around z_0.
- **Pole:** If z_0 is an isolated singularity as defined above and if $\lim_{z \to z_0} |f(z)| = \infty$ then z_0 is called a pole of f.
- **Essential singularity:** A singularity that is neither removable nor a pole.

Doubly infinite series

Let z_n be a function defined for $n = , 0, \pm 1, \pm 2, \cdots$, then it is doubly infinite.

- A doubly infinite series converges if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{=n}$ both converge.
- Splitting up the series in similar manners you can define absolute and uniform convergence.

Annulus

An annulus $A(z_0, R_1, R_2)$ around a point z_0 for $0 \leq R_1 \leq R_2$ is the set of all $z \in \mathbb{C}$ s.t. $R_1 \leq |z - z_0| \leq R_2$.

Laurent series expansion

Let f be a function holomorphic on $A(z_0, R_1, R_2)$, then there exists $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ s.t.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the doubly infinite series converges absolutely and uniformly in some $A(z_0, r_1, r_2)$ when $R_1 < r_1 < r_2 < R_2$.

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

where $\gamma(z) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $R_1 < r < R_2$.

Important results

- f has a removable singularity at $z_0 \iff a_n = 0$ for $n < 0$ in the Laurent series expansion of f
- f has a pole at z_0 of order $m \iff a_n = 0$ for $n < -m$ in the Laurent series expansion of f.
- f has a essential singularity at $z_0 \iff a_n \neq 0$ for infinitely many negative integers n.

Casorati–Weierstrass theorem

Let z_0 be an essential singularity of f then given $\alpha \in \mathbb{C}$, there exists a sequence $z_n \in B(z_0, R) \setminus \{z_0\}$ s.t. $z_n \to z_0$ and $f(z_n) \to \alpha$.

- Alternatively, f approaches any given value arbitrarily closely in any neighborhood of an essential singularity.

Meromorphic functions

Let Ω be a open connected subset of $\mathbb{C}$ and let $S \subset \Omega$. Let $f : \Omega \setminus S \to \mathbb{C}$ be holomorphic on Ω. We say that f is a meromorphic function on Ω if,

- S is a discrete set.
- f either has removable singularities or poles at point of S.

Operations on meromorphic functions

Let $\mathcal{M}(\Omega)$ denote the equivalence classes of meromorphic functions over Ω.

- We say that two meromorphic functions $f : \Omega \setminus S_1$ and $g : \Omega \setminus S_2$ are equivalent if $f(z) = g(z)$ on $\Omega \setminus (S_1 \bigcup S_2)$.
- For $f, g \in \mathcal{M}(\Omega)$, define $f + g$ to be the equivalence class of $(f + g) : \Omega \setminus (S_1 \bigcup S_2)$
- Similarly, fg is the equivalence class of $fg : \Omega \setminus (S_1 \bigcup S_2)$.

The space of all meromorphic functions is a field.

Order of meromorphic functions

The order of a meromorphic function is defined as follows,

- If $z_0 \in S$ is a removable singularity then the order of f at z_0 is the order of the zero at z_0 of f, i.e., $f(z) = (z - z_0)^m g(z)$ then m is the order.
- If $z_0 \in S$ is a pole and the pole is of order m then order of f at z_0 is $-m$.
- If $f \equiv 0$ then $\text{Ord}_{z_0} = \infty$.
- $\text{Ord}_{z_0}(f + g) \geq \min(\text{Ord}_{z_0}(f), \text{Ord}_{z_0}(g))$
- $\text{Ord}_{z_0}(fg) = \text{Ord}_{z_0}(f) + \text{Ord}_{z_0}(g)$

Residue of a function

Residue of a function: Let $f : \Omega \setminus S \to \mathbb{C}$ be a holomorphic function, where Ω is an open set and S is a discrete subset of Ω. Then for $z_0 \in S$, let $r > 0$ be s.t. $\overline{B(z_0, r)} \subseteq \Omega$ and $B(z_0, r) = \{z_0\}$. Then in $B(z_0, r) \setminus \{z_0\}$, consider the Laurent series expansion of f given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. We define the residue of f at z_0 to be $\text{Res}(f, z_0) = a_{-1}$.

1. If z_0 is a removable singularity then $\text{Res}(z_0) = 0$.
2. If z_0 is a pole of order m then $(z - z_0)^m f(z) = g(z)$, where $g(z) \neq 0$ on $B(z_0, r) \setminus \{z_0\}$then, $\text{Res}(z_0) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}$.

Residue theorem

Let Ω be an open connected subset of $\mathbb{C}$ and S be a finite subset of Ω and let $f : \Omega \setminus S \to \mathbb{C}$ be a holomorphic function. Let γ be a null homotopic closed curve on Ω. Then,

$$\frac{1}{2\pi i} \int_\gamma f(z) \, dz = \sum_{j=1}^{k} W_\gamma(z_j) \text{Res}(f, z_j)$$

where $S = \{z_1, \cdots, z_k\}$ and W_γ is the winding number.

Log derivative

For a holomorphic function $f : \Omega \to \mathbb{C}$. Define the log derivative of f to be the meromorphic function $\frac{f'(z)}{f(z)}$.

1. $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$
2. $\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}$
3. When f has a pole of order m at z_0 then for $f(z) = \frac{g(z)}{(z - z_0)^m}$ the log derivative of f is $\frac{g'(z)}{g(z)} - \frac{m}{(z - z_0)}$

Argument principle

Let $f : \Omega \setminus S \to \mathbb{C}$ be a meromorphic function s.t. f has zeros of order $d_1, \ldots, d_n$ at $z_1, \ldots z_n$ after removing the removable singularities. And f has poles of order $e_1, \ldots, e_m$ at points $w_1, \ldots, w_m$. Let γ be a closed curve which is null homotopic in Ω s.t. the zeros and poles don't lie in the image of γ. Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{i=0}^{n} d_i W_\gamma(z_i) - \sum_{j=1}^{m} e_j W_\gamma(w_j)$$

Rouche's theorem

Let γ be a closed curve which is null homotopic in Ω. Let f, g be functions holomorphic in Ω and $|g(z)| < |f(z)|$ on γ then f and $f+g$ have the same number of zeros counting multiplicities on the interior of $H([0, 1] \times [a, b])$ where H is the null homotopy from γ to a constant path.

Branch of the complex logarithm

Let Ω be an open connected subset of $\mathbb{C} \setminus \{0\}$. Define a branch of the logarithm on Ω as a function $f : \Omega \to \mathbb{C}$ s.t. $\exp(f(z)) = z, \forall z \in \Omega$.
For $\Omega = \mathbb{C} \setminus \{\Re(x) \leq 0\}$ define the standard branch to be

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z)$$

As defined above $\text{Log}(z)$ is holomorphic on Ω.

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Schwarz lemma

Let $\mathbb{D}$ denote the open unit disc. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomoprhic function s.t. $f(0) = 0$. Then,

$$|f(z)| \leq |z|, \forall z \in \mathbb{D}, \text{ and } |f'(z)| \leq 1$$

Also, if $|f(z)| = |z|$ for some $z \in \mathbb{D}$ or if $|f'(0)| = 1$ then $\exists \lambda \in \mathbb{C}, |\lambda| = 1$ s.t. $f(z) = \lambda z$.

Automorphism

A function $f : \Omega \to \Omega$ is an automorphism if f is holomorphic and has a holomorphic inverse.

Automorphisms of the unit disc

Define a function $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$ defined as $\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$.
Let $f : \mathbb{D} \to \mathbb{D}$ be an automorphism. Then there exists $\alpha \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ s.t.

$$f(z) = \lambda\varphi_\alpha(z)$$

Phragmén–Lindelöf method

Let $\Omega = \{z \in \Omega : a < \Re(z) < b\}$. Let $f : \overline{\Omega} \to \mathbb{C}$, s.t. f is continuous on $\overline{\Omega}$ and holomorphic on Ω. Suppose for some $z = x + iy$, we have $|f(z)| < B$ and let $M(x) = \sup\{|f(x + iy)| : -\infty < y < \infty\}$. Then,

$$M(x)^{b-a} \leq M(a)^{b-x}M(b)^{x-a}$$

And further

$$|f(z)| \leq M(x) \leq \max\{M(a), M(b)\} = \sup_{z\in\partial\Omega} |f(z)|$$

Schwarz-Pick theorem

First define $\rho(z, w) = \left|\frac{z-w}{1-\bar{w}z}\right|$ for $z, w \in \mathbb{D}$. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then,

$$\rho(f(z), f(w)) \leq \rho(z, w)\ \forall z, w \in \mathbb{D}$$

and,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}\ \forall z \in \mathbb{D}$$

Lifting of maps

Let X, Y, Z be open subsets of $\mathbb{C}$ and let $f : Y \to X$ and $g : Z \to X$ be continuous maps. Then we say, a map $\widetilde{g} : Z \to Y$ is a lift of g w.r.t. f if $f \circ \widetilde{g} = g$.

Uniqueness of lifts: Let X, Y, Z be open connected subsets of $\mathbb{C}$ and let $f : Y \to X$ be a *local* homeomorphism. Let $g : Z \to X$ be a continuous map. Let $\widetilde{g_1}$ and $\widetilde{g_2}$ be lifts of g w.r.t. f and suppose they are equal at some point in Z. Then $\widetilde{g_1} \equiv \widetilde{g_2}$.

- Let $f : Y \to X$ be a holomorphic map s.t. $f'(y) \neq 0$ on Y. Let $g : Z \to X$ be a holomorphic map s.t. $\widetilde{g} : Z \to Y$ is a lift of g w.r.t. f. Then $\widetilde{g}$ is holomorphic.
- Let X, Y be open subsets of $\mathbb{C}$ let, $f : Y \to X$ be a local homeomorphism. Let γ_0, γ_1 be curves in X from z_1 to z_2 which are homotopic. Suppose that for every $s \in [0, 1]$, we can lift $\gamma_s(t) = H(s, t)$ to a path $\widetilde{\gamma_s} : [a, b] \to Y$ w.r.t. f s.t. $\widetilde{\gamma_s}(a) = \widetilde{z_1}$, $\forall s \in [0, 1]$. Then $\widetilde{\gamma_0}, \widetilde{\gamma_1}$ are homotopic in Y.

Covering spaces

Let X, Y be open subsets of $\mathbb{C}$. We say that a continuous map $f : Y \to X$ is a covering map if given $x \in X$ there exists a neighbourhood U of X and open sets $\{V_\alpha\}_{\alpha\in A}$ in Y s.t. $f^{-1}(U) = \coprod_{\alpha\in A} V_\alpha$ (disjoint union of V_α) and $f|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Then Y is called a cover of X.

- Let $f : Y \to X$ be a covering map and $\gamma[a, b] \to X$ be a curve from x_0 to x_1 in X. Suppose $y_0 \in f^{-1}(\{x_0\})$. Then there exists a unique lift $\widetilde{\gamma}[a, b] \to Y$ of γ w.r.t. f s.t. $\widetilde{\gamma}(a) = y_0$.
- For connected X let $f : Y \to X$ be a covering map. Suppose $x_0, x_1 \in X$. Then the cardinality of $f^{-1}(x_0)$ is the same as the cardinality of $f^{-1}(x)$.
- For open subsets X, Y of $\mathbb{C}$ let, $f : Y \to X$ be a covering map from Y to X. Let Z be an open connected subset of $\mathbb{C}$, which is simply connected and locally connected. Suppose $g : Z \to X$ is a continuous map. Then given $z_0 \in C$ and $y_0 \in Y$ s.t. $g(z_0) = f(y_0)$, then there exists a unique lift $\widetilde{g} : Z \to Y$ of g w.r.t f.
- Let Ω be a simply connected, locally connected, open connected subset of $\mathbb{C}$ and $g : \Omega \to \mathbb{C}^*$ be a holomorphic map. Then there exists a lift $\widetilde{g} : \Omega \to \mathbb{C}$ s.t. $\exp(\widetilde{g}) = g$.

Bloch's theorem

- For $f : \mathbb{D} \to \mathbb{C}$ s.t. $f(0) = 0, f'(0) = 1$ and $|f(z)| \leq M\ \forall z \in \mathbb{D}$. Then $B(0, \frac{1}{6M}) \subseteq f(\mathbb{D})$.
- Let $f : B(0, R) \to \mathbb{C}$ be holomorphic s.t. $f(0) = 0, f'(0) = \mu$ for some $\mu > 0$ and $f|(z)| \leq M\ \forall z \in B(0, R)$. Then, $B(0, \frac{R^2\mu^2}{6M}) \subseteq f(B(0, R))$.

Bloch's theorem: Let Ω be an open connected subset of $\mathbb{C}$ s.t. $\overline{\mathbb{D}} \subset \Omega$. Let$f : \Omega \to \mathbb{C}$ s.t. $f(0) = 0, f'(0) = 1$. Then there exists a ball B' contained in $\mathbb{D}$ s.t. $f|_{B'}$ is injective and $B(0, \frac{1}{72}) \subseteq f(B') \subseteq f(\mathbb{D})$.

Little Picard's theorem

- Let Ω be an open connected subset of $\mathbb{C}$ which is simply connected. Let $f : \Omega \to \mathbb{C}$ which omits 0 and 1. Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$ s.t. $f(z) = -\exp(\pi i \cosh(2g(z)))$
- The function g as defined above doesn't contain any disk of radius 1.

Little Picard's theorem: If f is an entire function which omits two points, then f is a constant function.

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