Harold's Counting
Cheat Sheet

12 February 2025

(See also Harold's Sets Cheat Sheet)

Counting Rules

Rule	Description	Comments
Cardinality	The cardinality of A is the number of elements in set A = $\lvert A \rvert$	• if A = {(1,2), (3,4), (5,6)}, then |A| = 3 • Also denoted $n(A)$ • Cardinality = Counting
Product Rule	Let $A_1, A_2, \ldots, A_n$ be finite sets. Then, $\lvert A_1 \times A_2 \times \cdots \times A_n \rvert = \lvert A_1 \rvert \bullet \lvert A_2 \rvert \bullet \cdots \bullet \lvert A_n \rvert$	• Counts sequences • Think Intersection (∩)
Sum Rule	Consider n sets, $A_1, A_2, \ldots, A_n$. If the sets are mutually disjoint ($A_i \cap A_j = \emptyset$ for $i \neq j$), then $\lvert A_1 \cup A_2 \cup \ldots \cup A_n \rvert = \lvert A_1 \rvert + \lvert A_2 \rvert + \cdots + \lvert A_n \rvert$	• Counts sequences • Think Union (∪)
Generalized Product Rule	$\lvert S \rvert = n_1 \bullet n_2 \bullet \cdots \bullet n_k$ *n! = (n)(n-1)(n-2) … (2)(1)*	• In selecting an item from a set, if the number of choices at each step is independent, then the number of items in the set is the product of the number of choices in each step.
Bijection Rule	Let S and T be two finite sets. If there is a bijection from S to T, then $\lvert S \rvert = \lvert T \rvert$	• 1-to-1 Correspondence
k-to-1 Rule	$\lvert Y \rvert = \frac{\lvert X \rvert}{k}$	• k-to-1 Correspondence

Counting Formulas & Techniques

Rule	Description	Comments
Factorial	$n! = n \bullet (n-1) \bullet (n-2) \bullet \ldots \bullet 3 \bullet 2 \bullet 1$	• The number of permutations of a finite set with n elements is P(n,n)
Permutation	$P(n,r) = {}_nP_r = \frac{n!}{(n-r)!} = n(n-1)\ldots(n-r+1)$	• Order matters (,) • r-permutation • Counting sequences • Common application of the generalized product rule • Order can be fixed but arbitrary • Elements cannot be repeated • Use when elements are all <u>different</u>
Grouping	Password length is 18. No character repeats. Must contain: a, z, 1, and 9. $P(18,4) \bullet P(36-4, 18-4)$	• Combines permutation with product rule
Combination	$C(n,r) = {}_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$	• Order does not matter { , } • Counting subsets • r-combination • n choose r • Counting the r-subsets • Combination = subset • Use when elements are all <u>identical</u>
	Identity: $\binom{n}{n-r} = \binom{n}{r}$	• An equation is called an **identity** if the equation holds for all values for which the expressions in the equation are well defined.
Counting Subsets	Bijection from: 5-bit strings with exactly 2 1's To: 2-subsets of { 1, 2, 3, 4, 5 } = $\binom{5}{2} = 10$	• Binary example • Counting Strings by Counting Subsets
	$\binom{m+n}{m}$	• Counting paths on a grid {N, E} • Binary example if m = #1s & n = #0s

Counting with Discrete Probability

<table>
<tr><th>Rule</th><th>Formula</th><th>Definition</th></tr>
<tr><td>Notation</td><td>∩ = Intersection or “”and”
∪ = Union or “or”
$\overline{\ }$ = Negation or “not”</td><td>• “and” implies multiplication.
• “or” implies addition.
• “not” implies negation.</td></tr>
<tr><td>Independent</td><td>If $P(A|B) = P(A)$</td><td>• The occurrence of one event does not affect the probability of the other event, or vice versa.</td></tr>
<tr><td>Mutually Independent</td><td>No sets overlap.
Future outcomes are not impacted by previous outcomes.</td><td>• Applies to more than two events</td></tr>
<tr><td>Dependent</td><td>If $|A \cap B| \neq 0$</td><td>• The occurrence of one event affects the probability of the other event.</td></tr>
<tr><td>Disjoint
(“mutually exclusive”)</td><td>If $|A \cap B| = 0$, then
$|\boldsymbol{A} \cap \boldsymbol{B}| = |\boldsymbol{A}| + |\boldsymbol{B}|$</td><td>• The events can never occur together.</td></tr>
<tr><td>Probability
(“likelihood”)</td><td>$\boldsymbol{P(E)} = \frac{|\boldsymbol{E}|}{|\boldsymbol{S}|}$</td><td>• S = Sample space or entire set
• A, B, E = Event or subset

$0 \le P(E) \le 1$</td></tr>
<tr><td rowspan="4">Addition Rule
(“or”)</td><td>$|\boldsymbol{A} \cup \boldsymbol{B}|$
$= |A| + |B| - |A \cap B|$</td><td>Inclusion-Exclusion Principle
• Let A, B and C be three finite sets, then …
• If sets overlap, then don’t double count
• ”… in any of the 3.”
• ”… divisible by 2, 3,or 5.”</td></tr>
<tr><td colspan="2">$|\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}|$
$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$</td></tr>
<tr><td colspan="2">$|\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C} \cup \boldsymbol{D}|$
$= |A| + |B| + |C| + |D|$
$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$
$- |A \cap B \cap C \cap D|$</td></tr>
<tr><td>if mutually independent / disjoint:
$|A_1 \cup A_2 \cup \ldots \cup A_n| =$
$|A_1| + |A_2| + \cdots + |A_n|$</td><td>• A collection of sets is mutually disjoint if the intersection of every pair of sets in the collection is empty.
• Restatement of the Sum Rule</td></tr>
</table>

Multiplication Rule (“and”)	$\lvert A \cap B\rvert = \lvert A\rvert \cdot \lvert(B \mid A)\rvert$
	$\lvert A \cap B\rvert = \lvert B\rvert \cdot \lvert(A \mid B)\rvert$
	$\lvert A \cap B\rvert = \lvert A\rvert - \lvert A \cap \overline{B}\rvert$
	
	if independent / disjoint:
	$\lvert \boldsymbol{A} \cap \boldsymbol{B}\rvert = \lvert \boldsymbol{A}\rvert \cdot \lvert \boldsymbol{B}\rvert$
	
	if mutually independent / disjoint:
	$\lvert A \cap B \cap C\rvert = \lvert A\rvert \cdot \lvert B\rvert \cdot \lvert C\rvert$
	$\lvert A_1 \cap A_2 \cap \ldots \cap A_n\rvert$
	$= \lvert A_1\rvert \cdot \lvert A_2\rvert \cdot \ldots \cdot \lvert A_n\rvert$	

Complement Rule / Subtraction Rule (“not”)	$\boldsymbol{P}(\boldsymbol{S}) = P(E \cup \overline{E})$ $\lvert E\rvert + \lvert \overline{E}\rvert = \lvert S\rvert$ $\lvert E\rvert = \lvert S\rvert - \lvert \overline{E}\rvert$ $\lvert \overline{E}\rvert = \lvert S\rvert - \lvert E\rvert$ $\lvert(A \mid B)\rvert + \lvert(\overline{A} \mid B)\rvert = \lvert A\rvert$	• S = entire set, E = subset • The complement of event *E* (denoted $\overline{E}$ *or* E^c) means “**not *E***”; • It consists of all simple outcomes that are not in *E*. • ”has at least one” so choose $\overline{E}$ as “none”
Union by Compliment	$\lvert S\rvert - \lvert \overline{E_1 \cup E_2 \cup \ldots \cup E_n}\rvert =$ $\lvert E_1 \cup E_2 \cup \ldots \cup E_n\rvert$	• S = U = Universal set (all) • E.g., 10^4 - 9^4
Conditional Probability (“given that”)	$\boldsymbol{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \dfrac{\lvert \boldsymbol{A} \cap \boldsymbol{B}\rvert}{\lvert \boldsymbol{B}\rvert}$ if independent / disjoint: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = P(A)$ $\lvert(A \mid B)\rvert = \lvert A\rvert$ $\lvert(B \mid A)\rvert = \lvert B\rvert$	• Means the probability of event A given that event B has already occurred. • Is a rephrasing of the Multiplication Rule. • P(A|B) is the proportion of elements in B that are ALSO in A.
Total Probability Rule	$P(A) = P(A \cap B_1) + \cdots + P(A \cap B_n)$ $= P(B_1) \cdot P(A \mid B_1) + \cdots + P(B_n)$ $\cdot P(A \mid B_n)$ $P(A) = P(A \cap B) + P(A \cap \overline{B})$ $= P(A \mid B) \cdot P(B) + P(A \mid \overline{B}) \cdot P(\overline{B})$	• To find the probability of event A, partition the sample space into several disjoint events. • A must occur along with one and only one of the disjoint events.
Bayes’ Theorem	$P(A \mid B) = \dfrac{\lvert A \cap B\rvert}{\lvert B\rvert} = \dfrac{\lvert(B \mid A)\rvert \cdot \lvert A\rvert}{\lvert B\rvert}$ $= \dfrac{\lvert(\boldsymbol{B} \mid \boldsymbol{A})\rvert \cdot \lvert \boldsymbol{A}\rvert}{\lvert(\boldsymbol{B} \mid \boldsymbol{A})\rvert \cdot \lvert \boldsymbol{A}\rvert + \lvert(\boldsymbol{B} \mid \overline{\boldsymbol{A}})\rvert \cdot \lvert \overline{\boldsymbol{A}}\rvert}$	• Allows P(A|B) to be calculated from P(B|A). • Meaning it allows us to reverse the order of a conditional probability statement, and is the only generally valid method!

Sources:

- SNHU MAT 230 - Discrete Mathematics, zyBooks.