Harold's Differential Equation Models
Cheat Sheet

7 November 2026

Models

#	Model Name	Differential Equations	Solution / Key Results
1	**Exponential Growth and Decay**	$\frac{dP}{dt} = kP$	$P(t) = P_0 e^{kt}$ $A(t) = P_0 e^{rt}$
2	**Baseball Trajectory with Air Resistance**	$\frac{dv_x}{dt} = -\frac{k}{m} v_{rel}(v_x - u_x)$ $\frac{dv_y}{dt} = -g - \frac{k}{m} v_{rel}(v_y - u_y)$	$\vec{F}_D = -k\, v_{\text{rel}}\, (\vec{v} - \vec{v}_{air})$ $k = \frac{1}{2}\rho C_D A$ $v_{\text{rel}} = \lvert\vec{v} - \vec{v}_{\text{air}}\rvert$
3	**Newton's Law of Cooling / Heating**	$\frac{dT}{dt} = -k(T - T_a), k > 0$	$T(t) = T_a + (T_0 - T_a)e^{-kt}$ $k > 0$
4	**Logistic Equation (Verhulst)**	$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$	$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right) e^{-rt}}$
5	**Drug Concentration in Body**	$\frac{dA(t)}{dt} = -k_a A$ $\frac{dC}{dt} = \frac{F k_a}{V_d} A - k_e C$	$C(t) = B\left(e^{-k_a t} - e^{-k_e t}\right) + C_0 e^{-k_e t}$ $B = \frac{F k_a D}{V_d(k_e - k_a)}$ $A(0) = D,\ C(0) = C_0,\ k_a \neq k_e$
6	**Doomsday Differential Equation**	$\frac{dP}{dt} = kP(P - M)$ $k > 0,\ M > 0$	$P(t) = \frac{M}{1 - \left(1 - \frac{M}{P_0}\right) e^{kMt}}$
7	**Predator–Prey Model (Lotka–Volterra)**	$\frac{dR}{dt} = \alpha R - \beta RW$ $\frac{dW}{dt} = \delta RW - \gamma W$	Equilibria: $(R, W) = (0,0)$ $(R^*, W^*) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$
8	**SIR Epidemic Model**	$\frac{dS}{dt} = -\beta \frac{SI}{N}$ $\frac{dI}{dt} = \beta \frac{SI}{N} - \gamma I$ $\frac{dR}{dt} = \gamma I$ $\mathcal{R}_0 = \beta/\gamma$	Assuming $I(0) > 0$: • If $\mathcal{R}_0 S(0)/N > 1$, infections initially increase. • If $\mathcal{R}_0 S(0)/N = 1$, the infection's initial growth rate is zero. • If $\mathcal{R}_0 S(0)/N < 1$, infections initially decrease.
9	**Lorenz System**	$\frac{dx}{dt} = \sigma(y - x)$ $\frac{dy}{dt} = x(\rho - z) - y$ $\frac{dz}{dt} = xy - \beta z$	Equilibria: $E_0 = (0,0,0)$ $E_{\pm} = \left(\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1\right)$ For $\rho > 1$; ± signs match.

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1. Exponential Growth and Decay

Observations:

- Population (Exponential): A population of bacteria grows at a rate directly proportional to the current population size.
- Finance: With a fixed interest rate, a larger balance earns more interest per unit time.

Disciplines:

- Biology, Finance, Chemistry, Physics

Applications:

- Bacterial growth in a Petri dish ($+k$).
- Continuously compounded interest in a savings account.
- Radioactive decay (for example, uranium-238) $k < 0$.
  - Radioactive half-life relation $t_{1/2} = \ln 2 \,/|k|$.

Model:

$$\boxed{\frac{dP}{dt} \propto P}$$

$$\frac{dP}{dt} = kP$$

Separate variables and integrate:

$$\frac{dP}{P} = k\,dt$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln P = kt + c$$

Solve for $P(t)$:

$$e^{\ln P} = e^{kt+c}$$

$$P = e^{c}\, e^{kt} = Ce^{kt}$$

At $t = 0$ (initial condition):

$$P(0) = Ce^{k*0} = C \cdot 1 = C = P_0$$
$$P_0 > 0$$

Therefore

For population:

$$\boxed{P(t) = P_0 e^{kt}}$$

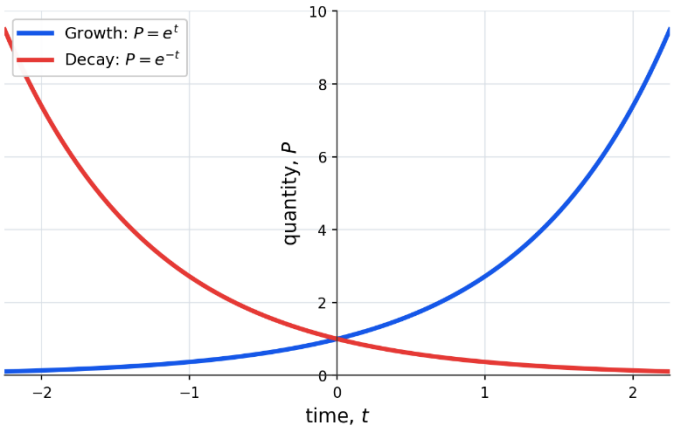
For finance:

$$\boxed{A(t) = P_0 e^{rt}}$$

Let:

- $\propto$: Symbol for "is proportional to". Replace "$\propto$" with "$= k$".
- $P(t)$: population or principal. $P(t) > 0$.
- $P_0 = P(0)$: initial population or principal.
- dP/dt: The instantaneous rate of change of P over time.
- t: time.
- k: constant (1/time).
- $A(t)$: account balance, continuous compounding.
- r: continuously compounded interest rate (1/time).

Graph:



Illustrative examples: $P_0 = 1$, $k = 1$ and $k = -1$.

Interpretation:

$k > 0$: Grows exponentially
$k = 0$: Constant
$k < 0$: Decays exponentially

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2. Baseball Trajectory with Air Resistance

Observations:

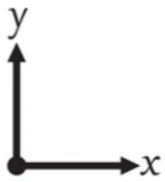
- A baseball follows a parabola only in the ideal no-drag model.
- Air resistance and wind change its path.

Disciplines:

- Classical mechanics; fluid dynamics

Application:

- Predicting a thrown baseball's landing location under wind and drag.



Model:

$$\vec{F}_D = -\frac{1}{2}\rho C_D A\,|\vec{v} - \vec{v}_{air}|\,(\vec{v} - \vec{v}_{air})$$

Equations of motion:

$$\frac{dx}{dt} = v_x, \qquad \frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = -\frac{k}{m} v_{rel}(v_x - u_x)$$

$$\frac{dv_y}{dt} = -g - \frac{k}{m} v_{rel}(v_y - u_y)$$

$$\vec{v}_{\text{air}} = (u_x, u_y)$$

$$v_{rel} = \sqrt{(v_x - u_x)^2 + (v_y - u_y)^2}$$

$$k = \frac{1}{2}\rho C_D A$$

Initial conditions:

$$x(0) = 0 \qquad v_x(0) = v_0 \cos\theta$$
$$y(0) = y_0 \qquad v_y(0) = v_0 \sin\theta$$

Find the landing point:

- Numerically integrate the equations from the release conditions.
- Find the descending crossing of ground level.
  - The horizontal position at that instant is the predicted landing distance.

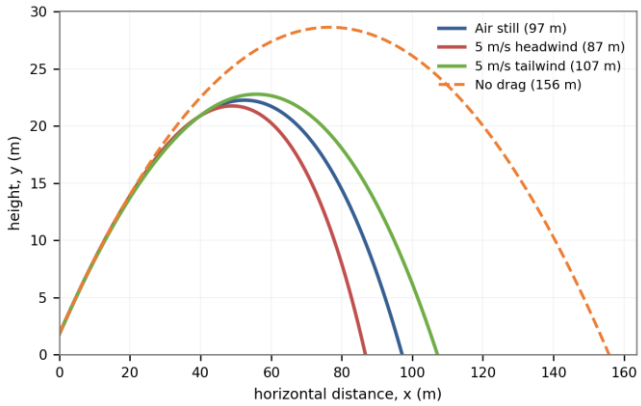
Accuracy notes:

- A reliable prediction needs the release speed and direction, release height, local wind, air density, and an appropriate drag coefficient.
- This 2-D example assumes head/tailwind.
- Crosswind needs a 3-D model.
- The ball's spin adds a Magnus force, and gusts or changing drag coefficients affect accuracy.

Let:

- m: baseball mass.
- A: cross-sectional area presented to the airflow.
- ρ: air density.
- C_D: drag coefficient.
- v: ball velocity.
  Wind velocity is subtracted to get relative velocity.
- v_{rel}: relative airspeed - the magnitude of the ball's velocity relative to the air.
- u_x: wind's horizontal component.
  Headwind: $u_x < 0$, Tailwind: $u_x > 0$
- u_y: wind's vertical component.
- k: combined drag factor; $g \approx 9.81 m/s^2$.
- v_0: launch speed.
- θ: launch angle (radians).
- y_0: release height. $y = 0$: ground level.

Graph of trajectories:



Illustrative inputs:

$$v_0 = 40 m/s$$
$$\theta = 35°$$
$$y_0 = 1.8 m$$
$$m = 0.145 kg$$
$$C_D = 0.35$$
$$\rho = 1.2 kg/m^3$$
$$u_x = -5\ m/s \text{ (headwind)}$$
$$A \approx 0.00419\ \text{m}^2$$

Steady wind is along the flight line ($u_y = 0$).

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3. Newton's Law of Cooling / Heating

Observation:

- The rate at which an object cools or heats is directly proportional to the temperature difference between the object and its surroundings.

Discipline:

- Thermal physics

Applications:

- Cooling/heating of objects.
- Coffee temperature, forensic time-of-death estimates.
- Electronics thermal modeling.

Model:

$$\boxed{\frac{dT}{dt} \propto \Delta T}$$

$$\Delta T = T_a - T$$

$$\frac{dT}{dt} \propto T_a - T$$

$$\frac{dT}{dt} = k(T_a - T)$$

$$\frac{dT}{dt} = -k(T - T_a)$$

Separate variables:

$$\frac{dT}{T - T_a} = -k\, dt$$

Integrate both sides:

$$\int \frac{1}{T - T_a} dT = \int -k\, dt$$

$$ln|T - T_a| = -kt - c$$

Solve for $T(t)$:

$$e^{ln|T-T_a|} = e^{-kt-c}$$

$$|T - T_a| = Ae^{-kt}, A > 0$$

$$T - T_a = \pm Ae^{-kt} = Ce^{-kt}$$

$$T(t) = T_a + Ce^{-kt}$$

where C is a signed constant

At $t = 0$ (initial condition):

$$T(0) = T_a + C = T_0$$

$$C = T_0 - T_a$$

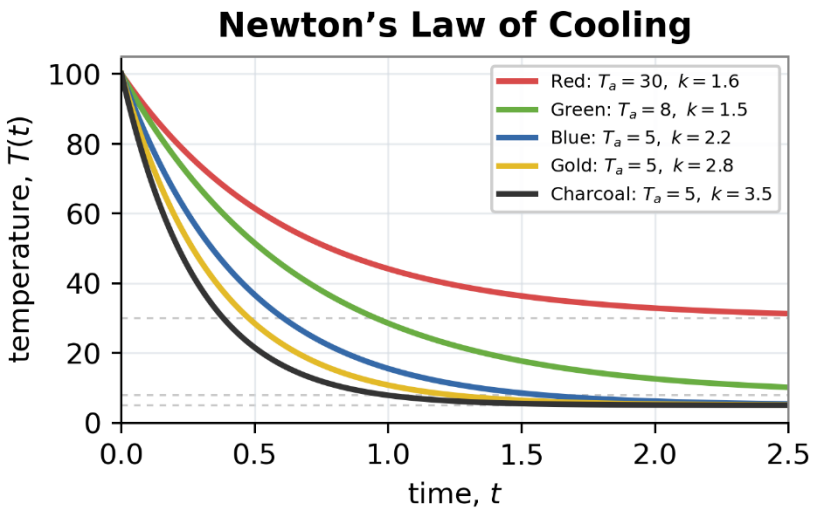
Therefore:

$$\boxed{\boldsymbol{T(t) = T_a + (T_0 - T_a)e^{-kt}}}$$

Let:

- $T(t)$: temperature (℃) of the object at time t.
- T_a: constant ambient temperature of the surroundings.
  - $T > T_a$: Cooling
  - $T < T_a$: Heating
- T_0: initial temperature of the object.
- k: positive constant (1/time).
  - Often referred to as the cooling constant.
  - It depends on the object's properties and the surrounding medium.

Cooling Graph:



$T_0 = 100$ for all curves; parameter estimates are illustrative.

Interpretation:

The object's temperature $T(t)$ approaches the ambient temperature T_a exponentially over time.

Equilibrium:

$T(t) = T_a$, occurs when $T_0 = T_a$.

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4. Logistic Equation (Verhulst)

Observation:

- A population's growth rate is proportional to its current size and to the fraction of the environment's carrying capacity that remains available.

Disciplines:

- Differential equations; population dynamics

Applications:

- Population growth with limited resources.
- Spread of ideas, product adoption, and some biological processes.

Model:

$$\boxed{\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)}$$

Separate variables:

$$\frac{dP}{P\left(1 - \frac{P}{K}\right)} = r\,dt$$

$$\frac{K}{P(K-P)}\,dP = r\,dt$$

Use partial fractions:

$$\left(\frac{1}{P} + \frac{1}{(K-P)}\right)dP = r\,dt$$

Integrate both sides:

$$ln|P| - ln|K-P| = rt + C_1$$

$$ln\left(\frac{P}{K-P}\right) = rt + C_1$$

Solve for $P(t)$:

$$\frac{P}{K-P} = e^{C_1}e^{rt} = C_2e^{rt}$$

$$C_2 = e^{C_1}, \qquad A = C_2^{-1}$$

$$P(t) = \frac{K}{1 + C_2^{-1}e^{-rt}} = \frac{K}{1 + Ae^{-rt}}$$

At $t = 0$ (initial condition):

$$P_0 = \frac{K}{1+A}$$

$$A = \frac{K - P_0}{P_0}$$

Therefore:

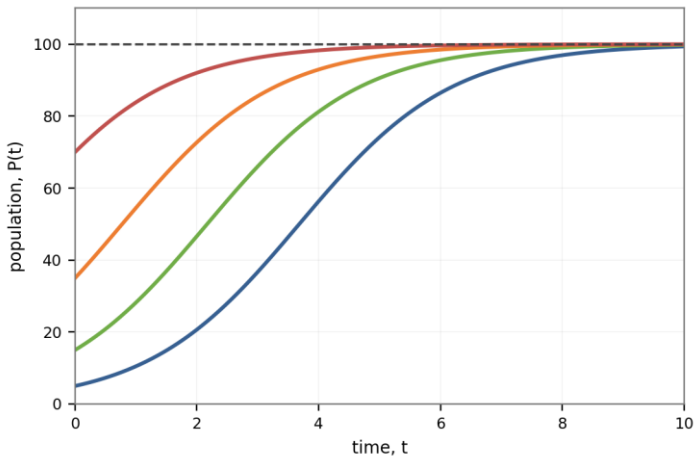
$$\boxed{\boldsymbol{P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}}}$$

Let:

- $P(t)$: population at time t.
- P_0: initial population, $P(0)$; assume $0 < P_0 < K$.
- K: fixed carrying capacity, the maximum population the environment can sustain.
- r: positive growth-rate constant (1/time)
- A: integration solution constant

$$A = \frac{K - P_0}{P_0}.$$

Illustrative Graph



Illustrative inputs:

$$K = 100$$

$$r = 0.8/time\ unit$$

$P_0 = 5,\ 15,\ 35,$ and $70,$ correspond to the blue, green, orange, and red curves, respectively.

The population approaches the carrying capacity K.
The growth rate $\frac{dP}{dt}$ is largest at $P = \frac{K}{2}$.

Equilibrium solutions:

$P(t) \equiv 0$ Extinct
$P(t) \equiv K$ Carrying-capacity equilibrium

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5. Drug Concentration in Body

Observations:

- The absorption of medicine into the body can be modeled using differential equations that describe how the concentration of the drug changes over time.
- A first-order absorption model assumes the rate of absorption is proportional to the amount of drug available at the absorption site (e.g., in the gastrointestinal tract (GI)).

1. Absorption phase by the GI tract:

The drug leaves the absorption site at a rate proportional to how much is left there.

$$\boxed{\frac{dA(t)}{dt} \propto A(t)}$$

$$\frac{dA(t)}{dt} = -k_a A(t)$$

$$A(t) = De^{-k_a t}$$

2. Entry into bloodstream:

The drug concentration in the bloodstream increases with absorption and decreases with elimination.

$$\frac{dC(t)}{dt} = Absorbed - Eliminated$$

$$\boxed{\frac{dC(t)}{dt} = \frac{F}{V_d} k_a A(t) - k_e C(t)}$$

Substitute $A(t)$ into the $C(t)$ differential equation:

$$\frac{dC(t)}{dt} = \frac{F}{V_d} k_a D e^{-k_a t} - k_e C(t)$$

Solving for $C(t)$ (see next page) gives the solution for the concentration in the bloodstream over time:

a) One oral dose with initial concentration C_0:

Case 1: $k_a \neq k_e$

$$A(0) = D$$
$$C(0) = C_0$$

$$\boxed{C(t) = \frac{F k_a D}{V_d(k_e - k_a)}\left(e^{-k_a t} - e^{-k_e t}\right) + C_0 e^{-k_e t}}$$

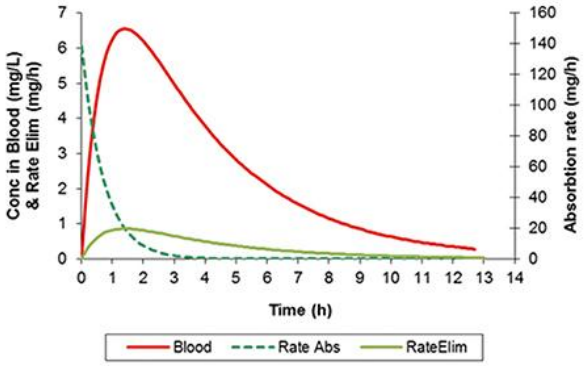
b) One oral dose with $C_0 = 0$:

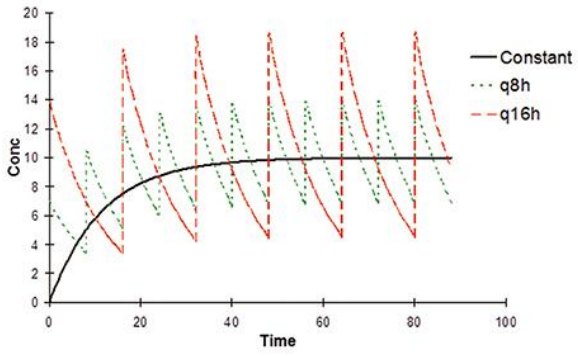
$$\boxed{C(t) = \frac{F k_a D}{V_d(k_e - k_a)}\left(e^{-k_a t} - e^{-k_e t}\right)}$$

Let:

- $A(t)$: amount of drug at the absorption site at time t.
- $C(t)$: drug concentration in the bloodstream (mg/L) at time t.
- C_0 : Existing concentration before the next dose.
- t : time (hours).
- D: dose of the drug in milligrams.
- k_a: rate constant for absorption of the drug (h^{-1}).
- k_e: rate constant for elimination of the drug (h^{-1}).
- F: bioavailability.
- V_d: volume of distribution (liters).

Graph:





The repeated-dose curve requires applying the dose interval by interval, updating C_0 after each dose.

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Case 2: $k_a = k_e = k$

$$\boxed{\boldsymbol{C(t) = e^{-kt}\left(C_0 + \frac{FkD}{V_d}t\right)}}$$

Derivation of the concentration formula:

Differential equation:

$$\frac{dC(t)}{dt} = \frac{F}{V_d}k_a D e^{-k_a t} - k_e C(t)$$

Move all C(t) variables to the left side:

$$\frac{dC(t)}{dt} + k_e C(t) = \frac{F}{V_d}k_a D e^{-k_a t}$$

Multiply both sides by the integrating factor, $e^{k_e t}$:

$$e^{k_e t}\frac{dC(t)}{dt} + k_e e^{k_e t} C(t) = \frac{F}{V_d}k_a D e^{(k_e - k_a)t}$$

The left-hand side is the product rule derivative of:

$$\frac{d}{dt}\left(e^{k_e t} C(t)\right) = \frac{F}{V_d}k_a D e^{(k_e - k_a)t}$$

Integrate:

$$\int \frac{d}{dt}\left[e^{k_e t} C(t)\right] dt = \int \frac{F}{V_d}k_a D e^{(k_e - k_a)t} dt$$

$$e^{k_e t} C(t) = \frac{F}{V_d}k_a D \int e^{(k_e - k_a)t}\, dt + K$$

Assume $k_a \neq k_e$ to avoid division by zero (0):

$$e^{k_e t} C(t) = \frac{F k_a D}{V_d(k_e - k_a)} e^{(k_e - k_a)t} + K$$

Solve for $C(t)$:

$$C(t) = \frac{F k_a D}{V_d(k_e - k_a)} e^{-k_a t} + K e^{-k_e t}$$

Apply the initial condition $C(0) = C_0$:

$$C_0 = \frac{F k_a D}{V_d(k_e - k_a)} + K$$

$$K = C_0 - \frac{F k_a D}{V_d(k_e - k_a)}$$

Therefore:

$$\boxed{\boldsymbol{C(t) = \frac{F k_a D}{V_d(k_e - k_a)}\left(e^{-k_a t} - e^{-k_e t}\right) + C_0 e^{-k_e t}}}$$

Graph interpretation:

- The first term is from absorption of the new dose.
- The second term is the decay of the existing concentration C_0 due to elimination.
- If $C_0 = 0$, we get the standard single-dose equation.

Assumptions:

- A one-compartment model with first-order absorption and elimination.
- For repeated oral doses, track both the amount awaiting absorption and the bloodstream concentration.
- At each dose, add D to A; the bloodstream concentration remains continuous.

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6. Doomsday Differential Equation

Observations:

- Population (Doomsday-threshold model): The "doomsday differential equation," or "doomsday-extinction model," is a differential equation that models population growth.
- It can lead to either population extinction or a "doomsday" time where the population explodes to infinity.

Model:

$$\boxed{\frac{dP}{dt} \propto P(P-M)}$$

$$\frac{dP}{dt} = kP(P-M)$$

$$k > 0,\ M > 0$$

Separate variables:

$$\frac{dP}{P(P-M)} = k\,dt$$

Partial fraction decomposition:

$$\frac{1}{P(P-M)} = \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right)$$

Substitute back in and multiply both sides by M:

$$\frac{1}{P-M}dP - \frac{1}{P}dP = kM\,dt$$

Integrate both sides:

$$\int \frac{1}{P-M}dP - \int \frac{1}{P}dP = \int kM\,dt$$

$$ln\,|P-M| - ln\,|P| = kMt + c$$

$$ln\left|\frac{P-M}{P}\right| = kMt + c$$

Solve for $P(t)$:

$$e^{ln\left|\frac{P-M}{P}\right|} = e^{kMt+c}$$

$$\left|\frac{P-M}{P}\right| = Ae^{kMt}$$

$$\frac{P-M}{P} = Ce^{kMt}, C \text{ is signed}$$

$$P(t) = \frac{M}{1-Ce^{kMt}}$$

At $t = 0$ (initial condition):

$$P(0) = P_0 \quad \longrightarrow \quad C = 1 - \frac{M}{P_0}$$

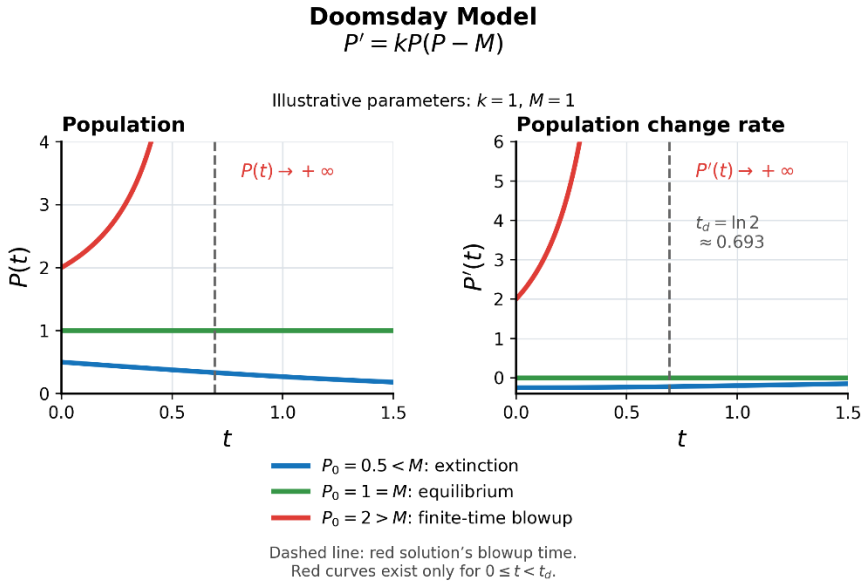
Therefore:

$$\boxed{\boldsymbol{P(t) = \frac{M}{1-\left(1-\frac{M}{P_0}\right)e^{kMt}}}}$$

Let:

- $P(t)$: population size at time t.
- P_0: initial population ($P_0 > 0$), where $P(0) = P_0$.
- k: growth rate ($k > 0$), units: $1/(\text{person} \cdot \text{time})$.
- M: threshold, units: population.

Graph:



Interpretation:

Initial population	Population $P(t)$	Rate $P'(t)$
$0 < P_0 < M$	Decreases toward 0 (extinction)	Negative; approaches 0 from below
$P_0 = M$	Constant at M	Zero
$P_0 > M$	Increases to $+\infty$ at finite time (doomsday)	Increases to $+\infty$ at the same time
	$t_{doomsday} = \left(\frac{1}{kM}\right)\ln\left(\frac{P_0}{P_0 - M}\right)$	

Equilibria:

$$P(t) \equiv 0$$
$$P(t) \equiv M$$

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7. Predator–Prey Model (Lotka–Volterra)

Observation:

- Predators depend on prey for food, while prey reproduce in the absence of predators.

Disciplines:

- Differential equations; mathematical ecology; population dynamics

Applications:

- Modeling and managing wolf and rabbit population dynamics.
- Comparing the effects of conservation, hunting, or relocation; these controls require added terms in the model.

Model:

$$\boxed{\begin{aligned}\frac{dR}{dt} &= \alpha R - \beta RW \\ \frac{dW}{dt} &= \delta RW - \gamma W\end{aligned}}$$

Nullclines (where population growth is zero):

$$\frac{dR}{dt} = 0: \quad R = 0 \text{ or } W = \frac{\alpha}{\beta}$$

$$\frac{dW}{dt} = 0: \quad W = 0 \text{ or } R = \frac{\gamma}{\delta}$$

Equilibrium populations:

$$(R, W) = (0,0)$$

$$(R^*, W^*) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$$

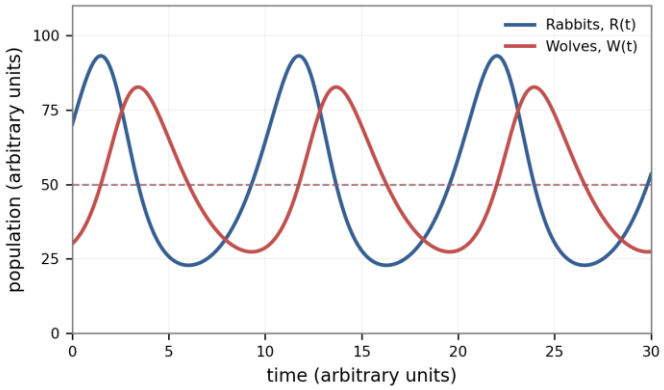
Interpretation:

- With no wolves, rabbits grow.
  With no rabbits, wolves decline.
- The populations cycle: rabbit peaks are followed by wolf peaks.
- In the classical model, cycles persist rather than damping out.
- Assumptions:
  - Constant rates
  - No migration
  - No resource limit on rabbit growth
- Real populations may show damped or irregular cycles.
- $(0,0)$ is a saddle.
- The positive coexistence equilibrium is a neutrally stable center.

Let:

- $R(t)$: rabbit (prey) population at time t.
- $W(t)$: wolf (predator) population at time t.
- α: rabbit per-capita growth rate without wolves (1/time).
- β: predation coefficient (1/(wolf·time)).
  Sets rabbit losses from predator encounters.
- γ: wolf per-capita death rate without rabbits (1/time).
- δ: conversion coefficient (1/(rabbit·time)).
  Links prey consumption to wolf growth.
- All four parameters are positive constants.

Graph of population cycles:



Illustrative inputs:

$$R(0) = 70$$
$$W(0) = 30$$
$$\alpha = 0.8$$
$$\beta = 0.016$$
$$\gamma = 0.5$$
$$\delta = 0.01$$

Equilibrium (R*, W*) = (50, 50) (dashed line)
Predator peaks lag prey peaks.

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8. SIR Epidemic Model

Observations:

- People move from susceptible to infectious, then to recovered or removed.
- The susceptible pool shrinks as infection spreads.

Disciplines:

- Epidemiology; differential equations; population dynamics

Application:

- Describing an outbreak, estimating its peak, and comparing vaccination or contact-reduction strategies.

Model:

$$\boxed{\begin{aligned}\frac{dS}{dt} &= -\beta\frac{SI}{N}\\ \frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I\\ \frac{dR}{dt} &= \gamma I\end{aligned}}$$

$$N = S + I + R = \text{constant}$$

Basic reproduction number:

$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

Initial infection grows if:

$$\mathcal{R}_0\frac{S(0)}{N} > 1$$

Infectious peak:

$$I(t) \text{ peaks when } S = \frac{N}{\mathcal{R}_0}$$

(when infections initially grow)

Interpretation:

Assuming $I(0) > 0$,

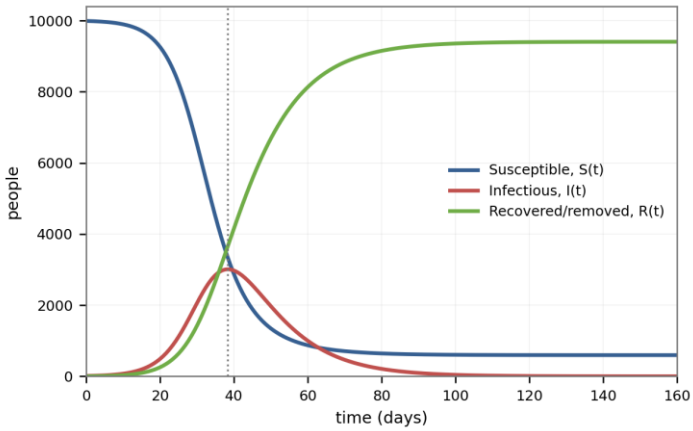
- If $\mathcal{R}_0 S(0)/N > 1$, infections initially increase.
- If $\mathcal{R}_0 S(0)/N = 1$, the infection's initial growth rate is zero $\left(\left.\frac{dI}{dt}\right|_{t=0} = 0\right)$.
- If $\mathcal{R}_0 S(0)/N < 1$, infections initially decrease.
- If infections initially increase, the infectious count peaks when $S = N/\mathcal{R}_0$. Otherwise, its maximum for $t \geq 0$ occurs at $t = 0$.

Assumptions: Closed population, homogeneous mixing, fixed transmission and recovery rates, and lasting immunity. R includes recovered or otherwise removed individuals (dead).

Let:

- $S(t)$: susceptible population.
- $I(t)$: infectious population.
- $R(t)$: recovered or removed population.
- N: constant total population.
- β: transmission-rate parameter (1/time).
- γ: recovery/removal rate (1/time); average infectious period in this model is $1/\gamma$.
- $\mathcal{R}_0$: basic reproduction number in the fully susceptible, homogeneous-mixing model.

Graph of an outbreak:



Illustrative inputs:

$$N = 10{,}000$$
$$\beta = 0.30/day$$
$$\gamma = 0.10/day$$
$$I(0) = 10$$
$$S(0) = 9{,}990$$
$$R(0) = 0$$
$$\text{Thus } \mathcal{R}_0 = 3$$

Curves are model estimates.

Vaccination lowers the susceptible pool; isolation or reduced contact lowers β.
These changes can reduce or prevent an outbreak.

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9. Lorenz System

Observations:

- A deterministic system can behave chaotically.
- Tiny changes in initial conditions can lead to very different long-term trajectories.

Disciplines:

- Chaos theory; nonlinear differential equations; atmospheric science

Applications:

- A simplified model of atmospheric convection.
- A classic example of sensitive dependence and chaotic motion.

Model:

$$\boxed{\begin{aligned}\frac{dx}{dt} &= \sigma(y - x)\\ \frac{dy}{dt} &= x(\rho - z) - y\\ \frac{dz}{dt} &= xy - \beta z\end{aligned}}$$

Equilibria:

$$E_0 = (0, 0, 0)$$

$$E_{\pm} = \left(\pm\sqrt{\beta(\rho - 1)}, \quad \pm\sqrt{\beta(\rho - 1)}, \quad \rho - 1\right)$$

$$\rho > 1$$

Classic chaotic parameter values:

$$\sigma = 10$$
$$\rho = 28$$
$$\beta = \frac{8}{3}$$

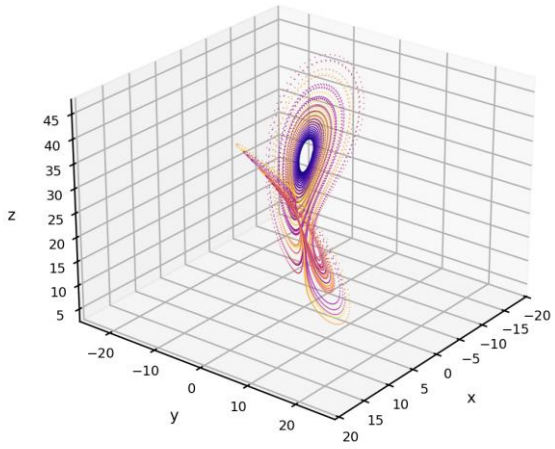
Interpretation:

- For $\rho > 1$, two nonzero equilibria exist; the ± signs in x and y match.
- For $\rho \leq 1$, the origin $(0, 0, 0)$ is the only equilibrium.
- At $\sigma = 10, \rho = 28, \beta = 8/3$, typical non-equilibrium trajectories approach a bounded chaotic attractor rather than settling to a fixed point.
- Chaos is deterministic, not random.
    - Long-term prediction is limited by sensitivity to initial conditions.

Let:

- x: scaled convective-flow intensity.
- y: scaled horizontal temperature variation.
- z: scaled vertical temperature variation.
- t: scaled model time
- σ: Prandtl number, a ratio of diffusivities.
- ρ: relative Rayleigh number, measuring thermal forcing.
- β: positive geometric parameter.

Graph of the Lorenz attractor:



Illustrative inputs:

Numerical trajectory for $\sigma = 10, \rho = 28, \beta = 8/3$, starting at $(0.1, 0.1, 0.1)$.

The first 5 model time units are omitted to show the attractor.

This low-order idealization illustrates convection and chaos; it is not a stand-alone weather forecast.

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