Harold's Ordinary Differential Equations (ODE) and Transforms Cheat Sheet

26 October 2026

Classification

Term	Definition
By Type	
Differential Equation (DiffEq)	A mathematical equation that relates a function to its derivatives. Example: $a_n(x)\,y^{(n)}(x) + a_{n-1}(x)\,y^{(n-1)}(x) + \cdots + a_1(x)\,y'(x) + a_0(x)\,y(x) = g(x)$
Ordinary Differential Equation (ODE)	A differential equation that involves an unknown function and its derivatives with respect to a single independent variable. Example: $\frac{dy}{dx} = x^2 + 3x$
Partial Differential Equation (PDE)	A type of equation that involves an unknown function and its partial derivatives with respect to multiple independent variables. Example: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$
Systems of Differential Equations	Two or more differential equations involving multiple unknown functions, solved together. Example: $\boldsymbol{x}' = A\boldsymbol{x}$
By Order	
Order	The order of the highest derivative in the equation. y' = 1st order y'' = 2nd order y''' = 3rd order $y^{(4)}$ = 4th order
By Linearity	
Linear Equation	$\boldsymbol{y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x)}$ $where\ each\ a_i(x)\ is\ a\ function\ of\ x.$
Nonlinear Equation	An equation that contains functions of y, such as e^y, or functions of the derivatives of y, such as $(5y)y'', sin(y'), or\ \left(\frac{dy}{dx}\right)^2$.
By Solution Type	
General Solution	A family of functions that has parameters and does not take any initial conditions into account.
Particular Solution	A specific solution with no arbitrary constants. An IVP solution also satisfies the given initial conditions.
Singular Solution	Cannot be obtained from the general solution.

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Notation and Terminology

Notation	Expanded Form	Description
Terms and Notation		
$\boldsymbol{x, y, z, t}$	x, y, z, t	Variables for space and time.
$\boldsymbol{f}$	$f = f(x)$ or y	The function of a single independent variable.
$\boldsymbol{f'}$	$\frac{df}{dx}, \frac{dy}{dx}, or\ y'$	The derivative of f with respect to x.
$\boldsymbol{f_x}$	$\frac{\partial f}{\partial x}$	The partial derivative of f with respect to x. All other variables are treated as constants.
$\boldsymbol{y_p}$	Particular solution to the equation.	
IVP	A differential equation together with initial conditions specified at the same point, so a particular solution can be found. $y(0) = 1, \quad y'(0) = -1$	

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Transforms

Symbol	Name	Description
$\mathcal{L}\{f(t)\}$	Laplace Transform	Converts linear ODEs with constant coefficients into algebraic equations incorporating initial conditions. Especially useful for discontinuous or impulsive forcing. $$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\ dt$$
$\mathcal{F}\{f(t)\}$	Fourier Transform	Transforms functions into the frequency domain. Useful for ODEs and PDEs on infinite or unbounded domains. Converts derivatives into algebraic factors. $$\mathcal{F}\{f(t)\} = \hat{f}(\omega) = \int_{-\infty}^\infty f(t)\, e^{-i\omega t}\ dt$$
$F_s(k)$ & $F_c(k)$	Fourier Sine and Cosine Transforms	Transform PDE boundary-value problems on $x \geq 0$. The sine transform naturally handles zero-value conditions at $x = 0$. The cosine transform naturally handles zero-slope conditions there. $$F_s(k) = \int_0^\infty f\,(x) \sin(kx)\ dx$$ $$F_c(k) = \int_0^\infty f\,(x) \cos(kx)\ dx$$
$\mathcal{H}_0\{f(r)\}(k)$	Hankel Transform (Order 0)	Transforms radially symmetric PDEs on unbounded radial domains in cylindrical or circular geometry. $$\mathcal{H}_0\{f(r)\}(k) = F(k) = \int_0^\infty f\,(r)\, J_0(kr)\, r\, dr$$ where J_0 is the Bessel function of the first kind, order zero.
$\mathcal{M}\{f(x)\}(s)$	Mellin Transform	Transforms equations with scaling symmetry and certain singular or asymptotic problems. $$\mathcal{M}\{f(x)\}(s) = F(s) = \int_0^\infty x^{s-1} f(x)\, dx.$$
$\mathcal{Z}\{x[n]\}$	Z-Transform	Transforms linear difference equations with constant coefficients into algebraic equations. $$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Conditions

- Integral transforms require convergence on their stated domains.

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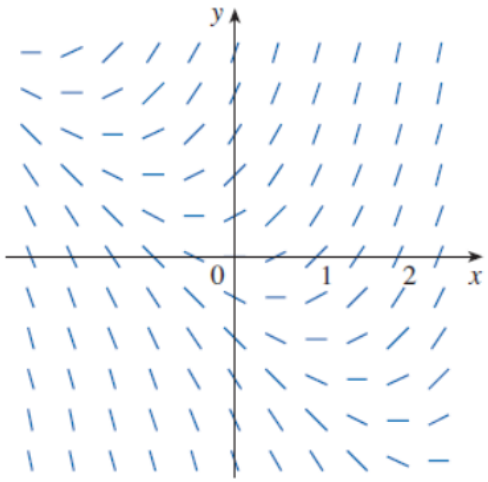
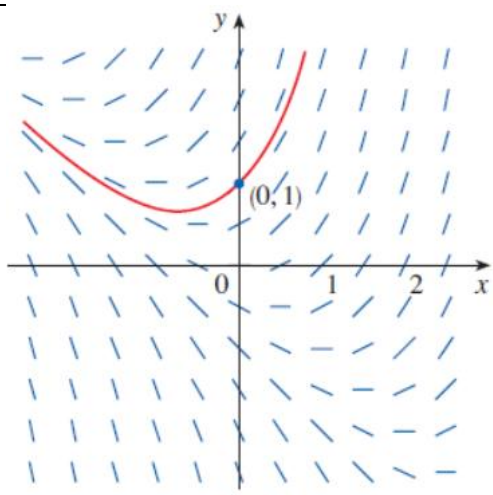
Direction Fields

Name	Slope Equation	Graph
Direction Field (Slope Field)	$y' = x + y$ **General** Solution: $y(x) = c_1 e^x - x - 1$	
Solution Curve	$\frac{dy}{dx} = x + y$ Initial Condition: $y(0) = 1$ **Particular** Solution: $y(x) = 2e^x - x - 1$	

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1st-Order Methods

Technique	Form of Equation	Solution Method
1st-Order Linear	$y' + a(x)y = f(x)$	$I.F. = e^{\int a(x)\,dx}$ $y = e^{-A(x)}\left[C + \int e^{A(x)} f(x)\,dx\right]$ $A(x) = \int a(x)\,dx$
Single Variable	$\frac{dy}{dx} = f(x)$	$y(x) = \int f(x)\,dx + C$
Separable Equations (Separation of Variables)	$\frac{dy}{dx} = f(x,y) = \frac{g(x)}{f(y)}$	$\int f(y)\,dy = \int g(x)\,dx + C$
Integrating Factors ($\mu(x)$)	$\frac{dy}{dx} + P(x)y = Q(x)$	1. $I.F. = \mu(x) = e^{\int P(x)\,dx}$ 2. Multiply the entire equation by $\mu(x)$. $\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$ 3. But the derivative of the product is: $\frac{d(\mu(x)y(x))}{dx} = \mu(x)\frac{dy}{dx} + \mu(x)P(x)y$ 4. $\mu(x)y(x) = \int \mu(x)\,Q(x)\,dx + C$ 5. General solution $\boldsymbol{y(x) = \frac{\int \mu(x)\,Q(x)\,dx + C}{\mu(x)}}$
Exact Equations	$M(x,y)\,dx + N(x,y)\,dy = 0$ $= d\Phi(x,y)$ $Iff \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Assume M, N have continuous partial derivatives on a simply connected domain.	Find a potential function $\Phi(x,y)$ such that $\Phi_x = M, \qquad \Phi_y = N.$ The implicit solution is $\Phi(x,y) = C$. Find it by integrating and comparing: $\int M(x,y)\,dx \quad and \quad \int N(x,y)\,dy$
Non-Exact Equations	Reduction to Exact via Integrating Factor. $\boldsymbol{\mu(x,y)\left[M(x,y) + N(x.y)\frac{dy}{dx}\right] = 0}$	
	If	**Then**
Case 1: $\mu(x)$	$h_1(x) \equiv \frac{M_y - N_x}{N}$	$\mu(x,y) = e^{\int h_1(x)\,dx}$
Case 2: $\mu(y)$	$h_2(y) \equiv \frac{N_x - M_y}{M}$	$\mu(x,y) = e^{\int h_2(y)\,dy}$
Case 3: $\mu(xy)$	$h_3(xy) \equiv \frac{N_x - M_y}{xM - yN}$	$\mu(x,y) = e^{\int h_3(xy)\,d(xy)}$

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Bernoulli	$\frac{dy}{dx} + P(x)y = Q(x)y^n$	If $n = 0$ or $n = 1$, this is linear; else 1. Divide by y^n 2. Change variable $z = y^{1-n}$ 3. Apply substitution to get a linear form 4. Use an integrating factor to solve 5. Check separately for solutions excluded by division or substitution, including $y = 0$ when applicable.
Homogeneous Substitution $(y = vx)$	$M(x,y)\,dx + N(x,y)\,dy = 0$ where $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree n, meaning: $M(\lambda x, \lambda y) = \lambda^n M(x,y)$ $N(\lambda x, \lambda y) = \lambda^n N(x,y)$	1. Start with: $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ 2. Reduce to separation of variables using: $v = \frac{y}{x}$ $y = vx$ 3. Product rule: $\frac{dy}{dx} = v + x\frac{dv}{dx}$ 4. Rewrite in terms of $\frac{dv}{dx}$.
Substitution	$\frac{dy}{dx} = F(ax + by + c)$ $b \neq 0$	1. Set $v = ax + by + c$ 2. Solve for y 3. Find $\frac{dy}{dx}$ in terms of $\frac{dv}{dx}$ 4. Rewrite the equation in terms of $\frac{dv}{dx}$
Reduction by Translation	$\boldsymbol{y' = \frac{Ax + By + C}{Dx + Ey + F}}$	
	If	**Then**
Case 1: Intersecting Lines	$AE - BD \neq 0$	1. Set $x = X + h,\ y = Y + k$ 2. Solve the system of equations: $Ah + Bk + C = 0$ $Dh + Ek + F = 0$ 3. Use $Y = vX$ 4. Solve using separation of variables 5. Translate back
Case 2: Parallel Lines	$AE - BD = 0$ $B \neq 0$	1. Put $u = Ax + By$ $y' = \frac{u' - A}{B}$ 2. Solve

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2nd-Order Linear Homogeneous ($= 0$)

Technique	Form of Equation	Solution Method
2nd-Order Linear	$P(x)y'' + Q(x)y' + R(x)y = \mathbf{0}$	The typical case has all constant (a, b, c) coefficients instead of functions of x.
Homogeneous	$ay'' + by' + cy = \mathbf{0}$	$y(x) = c_1 y_1(x) + c_2 y_2(x)$ where y_1, y_2 are linearly independent solutions.

Principle of Superposition	$y'' + ay' + by = L[y]$	**Case 1:**
		If $L[y_1] = 0$ and $L[y_2] = 0$, then
		$\boldsymbol{y = c_1 y_1 + c_2 y_2}$
		Case 2:
		If $L[y_1] = f_1$ and $L[y_2] = f_2$, then
		$\boldsymbol{L[y_1 + y_2] = f_1 + f_2}$

Characteristic Equation (Constant Coefficients)	$\boldsymbol{ay'' + by' + cy = 0}$ $\Downarrow$ $\boldsymbol{ar^2 + br + c = 0}$	Try a solution of the form: $y(x) = e^{rx}$ Roots: r_1, r_2
	If	**Then**
Case 1: Distinct Roots	$r_1, r_2 \in \mathbb{R}$ $r_1 \neq r_2$	1. Set $y_1(x) = e^{r_1 x}$ $y_2(x) = e^{r_2 x}$ 2. Superposition gives $\boldsymbol{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$
Case 2: Repeated Root	$r_1, r_2 \in \mathbb{R}$ $r_1 = r_2 = r$	$\boldsymbol{y(x) = c_1 e^{rx} + c_2 x e^{rx}}$
Case 3: Complex Roots	$r_{1,2} = \lambda \pm \mu\, i$ with $\lambda = -\frac{b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}$	1. Set $y_1(x) = e^{(\lambda + \mu\, i)x}$ $y_2(x) = e^{(\lambda - \mu\, i)x}$ 2. Superposition gives $y(x) = c_1 e^{(\lambda + \mu\, i)x} + c_2 e^{(\lambda - \mu\, i)x}$ 3. Apply $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ 4. $\boldsymbol{y(x) = c_1 e^{\lambda x} \cos(\mu x) + c_2 e^{\lambda x} \sin(\mu x)}$
Reduction of Order	$P(x)y'' + Q(x)y' + R(x)y = 0$	If we already know y_1, 1. Put $y_2 = v y_1$ 2. Expand in terms of v'', v', v 3. Put $z = v'$ 4. Solve the reduced equation
Wronskian (Linear Independence)	For two solutions of the same regular 2nd-order linear homogeneous ODE $[y_1(x)\ and\ y_2(x)]$, linear independence is equivalent to $W(x_0) \neq 0$ at some point in the interval.	$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

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Euler-Cauchy Equation	$x^2y'' + axy' + by = 0$ where $x > 0$ $A.E.: \lambda(\lambda - 1) + a\lambda + b = 0$	1. $y(x)$ of the form x^λ 2. Reduction to Constant Coefficients: Use $x = e^t, t = \ln x$ 3. Rewrite in terms of t using the chain rule.
Case 1: Distinct Roots	$\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 \neq \lambda_2$ $x > 0$	$y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$
Case 2: Repeated Root	$\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 = \lambda_2 = \lambda$ $x > 0$	$y(x) = x^\lambda (c_1 + c_2 \ln\|x\|)$
Case 3: Complex Roots	$\lambda_{1,2} = \alpha \pm i\beta$ $x > 0$	$y(x) = x^\alpha (c_1 \cos(\beta \ln\|x\|) + c_2 \sin(\beta \ln\|x\|))$

2nd-Order Linear Non-Homogeneous: Variation of Parameters ($= g(x)$)

Technique	Form of Equation	Solution Method
2nd-Order Linear	$y'' + P(x)y' + Q(x)y = g(x)$	Works beyond constant-coefficient equations when the homogeneous fundamental solutions are known.
Variation of Parameters	$y'' + P(x)y' + Q(x)y = g(x)$	$y_p = -y_1 \int \frac{y_2 g}{W}\, dx + y_2 \int \frac{y_1 g}{W}\, dx$ $W = y_1 y_2{}' - y_1{}' y_2$
General Solution		$y = c_1 y_1 + c_2 y_2 + y_p$

Conditions

- y_1, y_2 are linearly independent solutions of the associated homogeneous equation.
- $W \neq 0$ on the working interval.

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2nd-Order Methods

Technique	Form of Equation	Solution Method
2nd-Order Linear	$y'' + a(x)y' + b(x)y = \boldsymbol{f(x)}$	$y = y_h + y_p$ $= \boldsymbol{c_1 y_1(x) + c_2 y_2(x) + y_p(x)}$ where y_1, y_2 are linearly independent solutions, and the general solution includes the particular solution, y_p.
Non-Homogeneous*	$P(x)y'' + Q(x)y' + R(x)y = \boldsymbol{f(x)}$ $ay'' + by' + cy = \boldsymbol{f(x)}$ $f(x) \not\equiv 0$	If homogeneous, then $f(x) = 0$.
Case 1: y', y Missing	$y'' = f(x)$	$y(x) = \int \left(\int f(x)\, dx \right) dx + C_1 x + C_2$
Case 2: y Missing	$y'' = f(y', x)$	1. Change of variable: $p = y'$ 2. Solve the 1st-order equation for p. 3. Then solve $y' = p$
Case 3: x Missing	$y'' = f(y', y)$	1. Change of variable: $p = y'$ + chain rule 2. $p\frac{dp}{dy} = f(p, y)$ is a 1st-order ODE 3. Solve it 4. Back-replace p 5. Solve again
Case 4: y', x Missing	$y'' = f(y)$	1. Change of variable: $p = y'$ + chain rule 2. $p\frac{dp}{dy} = f(y)$ is a separation of vars 3. Solve it 4. Back-substitute p 5. Solve again
Undetermined Coefficients (Educated Guesswork)	Choose a trial particular solution $y_p(x)$ based on the form of $f(x)$. Valid forms: 1. $P_n(x)$ 2. $P_n(x)\, e^{ax}$ 3. $e^{ax}\,(P_n(x)\cos bx + Q_n(x)\sin bx)$ <u>Procedure</u>: • Choose a trial solution containing all terms of the appropriate polynomial, exponential, sine, or cosine form. • For each trial group that overlaps the homogeneous solution, multiply that group by x^m, where m is the multiplicity of the corresponding characteristic root. • Substitute into the ODE and solve for the unknown coefficients. NOTE: This method applies to linear ODEs with constant coefficients.	

*Cases 2-4 can be nonlinear.

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Laplace Transforms

Technique	Equations and Descriptions
Laplace Transform	$\mathcal{L}\{f(t)\} = F(s)$ • An integral transform that converts a function of a real variable (usually t, in the time domain) to a function of a complex variable s in the complex-valued frequency domain (s domain). ○ NOTE: Frequency $s = \sigma + i\omega$ is a complex transform variable with units of inverse time. • Transforms ODEs into algebraic equations. • The z-transform is the discrete analog of the Laplace transform in signal processing.
Inverse Laplace Transform	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
Laplace Transform Equation	$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\ dt$ where • $f(t)$ must be defined on $0 \le t \le \infty$ • s lies in the region of convergence; for real s, take s sufficiently large. • The improper integral must converge
Transform of a Derivative	**Theorem**: If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on [0, ∞) and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on [0, ∞), then the Laplace transform of the nth derivative of $f(t)$ is given by the general nth order derivative equation below.
Laplace Transform of Derivatives	1st order derivative: $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$ 2nd order derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$ 3rd order derivative: $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - s\,f'(0) - f''(0)$ nth order derivative: $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
Using Laplace to Solve a DiffEq	Solves IVPs for linear differential equations with constant coefficients. 1. Apply the Laplace transform to each term in the IVP. 2. Solve the resulting algebraic equation for the transformed function $F(s)$. 3. Apply the inverse Laplace transform to find the solution.

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Table of Laplace Transforms

#	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	Restrictions
1	$f'(t)$	$sF(s) - f(0)$	
2	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
3	$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
4	$f^{(4)}(t)$	$s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$	
5	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	
6	1	$\frac{1}{s}$	
7	e^{at}	$\frac{1}{s-a}, s > a$	$s > a$
8	te^{at}	$\frac{1}{(s-a)^2}$	$s > a$
9	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$	$s > a$ $n = 1, 2, 3, \ldots$
10	$\frac{e^{at} - e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$	$a \neq b$ $s > \max(a, b)$
11	$\frac{ae^{at} - be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$	$a \neq b$ $s > \max(a, b)$
12	t	$\frac{1}{s^2}$	
13	t^n	$\frac{n!}{s^{n+1}}$	$n = 1, 2, 3, \ldots$
14	t^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$p \in \mathbb{R}, p > -1$
15	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	
16	$t^{n-\frac{1}{2}}$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$	$n = 1, 2, 3, \ldots$
17	$sin(at)$	$\frac{a}{s^2 + a^2}$	
18	$cos(at)$	$\frac{s}{s^2 + a^2}$	
19	$sin(at + b)$	$\frac{s\,sin(b) + a\,cos(b)}{s^2 + a^2}$	
20	$cos(at + b)$	$\frac{s\,cos(b) - a\,sin(b)}{s^2 + a^2}$	
21	$t\,sin(at)$	$\frac{2as}{(s^2 + a^2)^2}$	
22	$t\,cos(at)$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	
23	$\frac{1}{t}\sin(at)$	$\arctan\left(\frac{a}{s}\right)$	
24	$e^{at}\,sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$

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25	$e^{at}\,cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
26	$sin(at) - at\,cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	
27	$sin(at) + at\,cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$	
28	$cos(at) - at\,sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	
29	$cos(at) + at\,sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$	
30	$sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$
31	$cosh(at)$	$\frac{s}{s^2-a^2}, s > \lvert a\rvert$	$s > \lvert a\rvert$
32	$t\,sinh(at)$	$\frac{2as}{(s^2-a^2)^2}$	$s > \lvert a\rvert$
33	$t\,cosh(at)$	$\frac{s^2+a^2}{(s^2-a^2)^2}$	$s > \lvert a\rvert$
34	$e^{at}\,sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	$s > a + \lvert b\rvert$
35	$e^{at}\,cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$	$s > a + \lvert b\rvert$
36	$e^{at}f(t)$	$F(s-a)$	
37	$tf(t)$	$-F'(s)$	
38	$t^n\,f(t)$	$(-1)^n\,F^{(n)}(s)$	$n = 1, 2, 3, \ldots$
39	$\frac{1}{t}\,f(t)$	$\int_s^\infty F(u)\,du$ (Requires convergence of both sides, including integrability near $t = 0$.)	
40	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), c > 0$	
41	$u_a(t) = u(t-a)$ Heaviside Function	$\frac{1}{s}e^{-as}$	Nonnegative delay $a \geq 0$
42	$\delta(t)$	1	
43	$\delta(t-a)$ Dirac Delta Function	e^{-as}	Nonnegative delay $a \geq 0$. (Using $a > 0$ for the shifted impulse avoids the endpoint convention.)
44	$u_a(t)\,f(t-a)$	$e^{-as}\,F(s)$	
45	$u_a(t)\,g(t)$	$e^{-as}\,\mathcal{L}\{g(t+a)\}$	
46	$\int_0^t f(v)\,dv$	$\frac{1}{s}F(s)$	
47	$\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$	

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48	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$	$T>0$, with the periodic function integrable over one period
49	$af(t)+bg(t)$	$aF(s)+bG(s)$	
50	$\frac{1}{\sqrt{\pi t}} e^{\frac{-a^2}{4t}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$a>0$
51	$\frac{a}{2\sqrt{\pi t^3}} e^{\frac{-a^2}{4t}}$	$e^{-a\sqrt{s}}$	$a>0$
52	$erfc\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$	$a>0$

Conditions

- Valid in the region of convergence, typically $s>0$.
- Γ is the Gamma function, which extends factorials beyond nonnegative integers. For n = 0, 1, 2, …, $\Gamma(n+1)=n!$.

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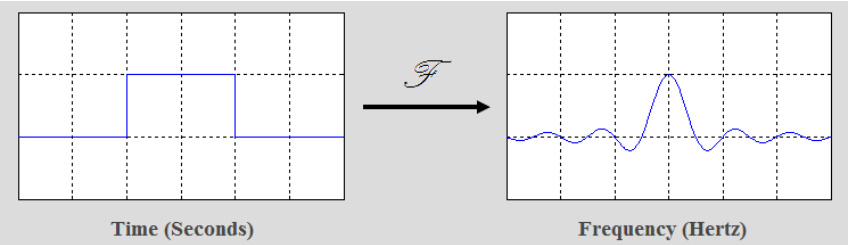
Fourier Transforms

Technique	Equations and Descriptions
Fourier Series	• A method to **decompose** a periodic signal into a sum of sine and cosine functions with different frequencies and amplitudes. • Is particularly useful for analyzing arbitrary **periodic** signals. • Was introduced by Joseph Fourier. • Is widely used in fields such as electronics, signal processing, and quantum mechanics.
Fourier Transform	• A mathematical operation that **transforms** a signal from the time domain (t) to the frequency domain (ω). • Unlike the Fourier Series, the Fourier Transform can be applied to both **periodic** and **non-periodic** signals. • Is widely used to solve differential equations. 
Fourier Transform Equation	$$\mathcal{F}\{f(t)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$ where • $f(t)$ must be defined on $-\infty \leq t \leq \infty$ • ω is an arbitrary real variable where $\omega = 2\pi f$ • The improper integral must converge
Inverse Fourier Transform Equation	$$\mathcal{F}^{-1}\{\hat{f}(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$$ where • $\hat{f}(\omega)$ must be defined on $-\infty \leq \omega \leq \infty$ • ω is an arbitrary real variable where $\omega = 2\pi f$ • The improper integral must converge Under standard inversion conditions, the inverse gives $f(t)$ at continuity points and the average of the one-sided limits at a jump.
Using Fourier Transforms to Solve a DiffEq	Solves ODEs for linear differential equations with constant coefficients. 1. Apply the Fourier transform to each term in the ODE. 2. Solve the resulting algebraic equation for the transformed function $\hat{f}(\omega)$. 3. Apply the inverse Fourier transform to find the solution.
Fourier Qualifications	• Ordinary transforms use convergent integrals. • Entries involving constants, sinusoids, polynomials, or delta functions are interpreted as distributions. • For $1/x$, use principal value; for $1/x^n, n \geq 2$, use finite part.

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Example	$$f(t) = \begin{cases} e^{-at} & for\ t > 0 \\ 0 & for\ t < 0 \end{cases}$$ $a > 0$ for real a $$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-it\omega}\, dt = \frac{1}{a + i\omega}$$

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Table of Fourier Transforms ($a, b > 0$)

#	$f(x)$	$\mathcal{F}\{f(x)\} = \hat{f}(\omega)$
1	$a\,f(x) + b\,g(x)$	$a\,\hat{f}(\omega) + b\,\hat{g}(\omega)$
2	$f(x-a)$	$e^{-ia\omega}\,\hat{f}(\omega)$
3	$f(x)\,e^{iax}$	$\hat{f}(\omega - a)$
4	$f(ax)$	$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{\omega}{a}\right)$
5	$\frac{d^n f(x)}{dx^n}$	$(i\omega)^n \hat{f}(\omega)$
6	$x^n f(x)$	$i^n\,\frac{d^n \hat{f}(\omega)}{d\omega^n}$
7	$(f * g)(x)$	$\hat{f}(\omega)\,\hat{g}(\omega)$
8	$f(x)\,g(x)$	$\frac{1}{2\pi}\,(\hat{f} * \hat{g})(\omega)$
9	$Real\,f(x)$	$\hat{f}(-\omega) = \overline{\hat{f}(\omega)}$
10	$Imaginary\,f(x)$	$\hat{f}(-\omega) = -\overline{\hat{f}(\omega)}$
11	$\overline{f(x)}$	$\overline{\hat{f}(-\omega)}$
12	$f(x)\,cos(ax)$	$\frac{\hat{f}(\omega - a) + \hat{f}(\omega + a)}{2}$
13	$f(x)\,sin(ax)$	$\frac{\hat{f}(\omega - a) - \hat{f}(\omega + a)}{2i}$
14	$e^{-ax}\,u(x)$	$\frac{1}{a + i\omega}$
15	$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2 + \omega^2}$
16	1	$2\pi\delta(\omega)$
17	$\delta(x)$	1
18	$sin(ax)$	$-i\pi(\delta(\omega - a) - \delta(\omega + a))$
19	$cos(ax)$	$\pi(\delta(\omega - a) + \delta(\omega + a))$
20	$sin(ax^2)$	$-\sqrt{\frac{\pi}{a}}sin\left(\frac{\omega^2}{4a} - \frac{\pi}{4}\right), a > 0$
21	$cos(ax^2)$	$\sqrt{\frac{\pi}{a}}cos\left(\frac{\omega^2}{4a} - \frac{\pi}{4}\right)$
22	$sech(ax)$	$\frac{\pi}{a}sech\left(\frac{\pi}{2a}\omega\right)$
23	$e^{-\pi i a x^2}$	$\frac{1}{\sqrt{\alpha}}\,e^{-\frac{i\pi}{4}}\,e^{\frac{i\omega^2}{4\pi\alpha}}, \alpha > 0$
24	x^n	$2\pi i^n\,\delta^{(n)}(\omega)$
25	$\delta^{(n)}(x)$	$(i\omega)^n$
26	$\frac{1}{x}$	$-i\pi\,sgn(\omega)$
27	$\frac{1}{x^n}$	$-i\pi\frac{(-i\omega)^{n-1}}{(n-1)!}sgn(\omega)$
28	$\frac{1}{\sqrt{\lvert x\rvert}}$	$\frac{\sqrt{2\pi}}{\sqrt{\lvert\omega\rvert}}$

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Z-Transforms

Technique	Equations and Descriptions
Z-Transform	$$\mathcal{Z}\{x[n]\} = X(z)$$ • Converts a discrete sequence $x[n]$ into a function $X(z)$ of a complex variable z. • Transforms linear difference equations with constant coefficients into algebraic equations. • Used for recurrences, sampled signals, and digital systems. • This sheet uses the one-sided (unilateral) Z-transform, with $n = 0,1,2, \ldots$. • The region of convergence (ROC) consists of the values of z for which the series converges absolutely.
Inverse Z-Transform	$$\mathcal{Z}^{-1}\{X(z)\} = x[n]$$ • Use a transform table, partial fractions, or expansion in powers of z^{-1}. • In the expansion, the coefficient of z^{-n} is $x[n]$.
Z-Transform Equation	$$\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$ where • n: integer sample index with $n = 0,1,2, \ldots$. • z: dimensionless complex transform variable. • $u[n]$: unit step; equals 1 for $n \geq 0$ and 0 otherwise. • $\delta[n]$: unit impulse; equals 1 at $n = 0$ and 0 otherwise.
Using Z-Transforms to Solve a Difference Equation	1. Apply the Z-transform to each term. 2. Use the shift formulas and substitute the initial values. 3. Solve the resulting algebraic equation for $Y(z)$. 4. Apply the inverse Z-transform to find $y[n]$.
Connection to Laplace Transforms	• For sampling interval T, the substitution $z = e^{sT}$ gives $z^{-n} = e^{-snT}$. • This relates the transform kernels; a Z-transform is not generally equal to the Laplace transform of the original continuous-time signal.
Connection to Fourier Transforms	• On the unit circle, $z = e^{i\Omega}$, the Z-transform equals the discrete-time Fourier transform when the unit circle lies in the ROC. • Here, Ω is the angular frequency in radians per sample.

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"Rabbit Growth" Example

Let $p[n]$ count rabbit pairs at year n, assuming no deaths and that each pair begins producing one new pair per year after a one-year maturation period.

Difference Equation:

$$p[n+2] = p[n+1] + p[n], \qquad p[0] = 1, \quad p[1] = 2.$$

Define $P(z) = \mathcal{Z}\{p[n]\}$. Applying the shift formulas:

$$z^2P(z) - z^2p[0] - zp[1] = zP(z) - zp[0] + P(z).$$

Therefore,

$$P(z) = \frac{z^2 + z}{z^2 - z - 1}.$$

Taking the inverse transform:

$$p[n] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right].$$

The sequence $1, 2, 3, 5, 8, 13, \ldots$ is the Fibonacci sequence starting at F_2.

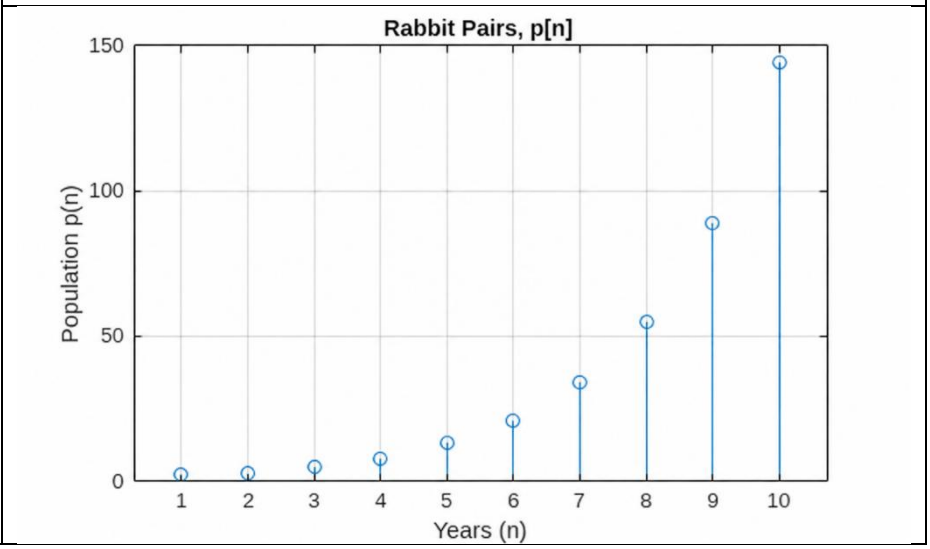

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Table of One-Sided Z-Transforms

#	Sequence $x[n]$	*Z-Transform* $X(z)$	Region of Convergence
1	$\delta[n]$	1	*All z*
2	$\delta[n-m]$ integer $m \geq 1$	z^{-m}	$z \neq 0$
3	$u[n]$	$\frac{z}{z-1}$	$\lvert z \rvert > 1$
4	a^n	$\frac{z}{z-a}$	$\lvert z \rvert > \lvert a \rvert$
5	n	$\frac{z}{(z-1)^2}$	$\lvert z \rvert > 1$
6	n^2	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z \rvert > 1$
7	n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z \rvert > 1$
8	na^n	$\frac{az}{(z-a)^2}$	$\lvert z \rvert > \lvert a \rvert$
9	n^2a^n	$\frac{az(z+a)}{(z-a)^3}$	$\lvert z \rvert > \lvert a \rvert$
10	$\cos(n\theta)$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$	$\lvert z \rvert > 1$
11	$\sin(n\theta)$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$	$\lvert z \rvert > 1$
12	$a^n\cos(n\theta)$	$\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$	$\lvert z \rvert > \lvert a \rvert$
13	$a^n\sin(n\theta)$	$\frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$	$\lvert z \rvert > \lvert a \rvert$
14	$u[n]-u[n-N]$ Integer $N \geq 2$	$\sum_{k=0}^{N-1} z^{-k} = \frac{1-z^{-N}}{1-z^{-1}}$	$z \neq 0$ value at $z=1$ is N

Conditions

- All sequences are evaluated for $n \geq 0$.
- Let $X(z) = \mathcal{Z}\{x[n]\}, Y(z) = \mathcal{Z}\{y[n]\}$, $a \neq 0$, and θ be real.
- If θ is an integer multiple of π, the sine sequence is identically zero, so its transform is zero for all z.

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Properties and Shift Formulas for Z-Transforms

#	Property	Sequence / Operation	Z-Transform
15	Linearity	$Ax[n] + By[n]$	$AX(z) + BY(z)$
16	Delay by m samples	$x[n-m]\,u[n-m]$	$z^{-m}X(z)$
17	Advance by one sample	$x[n+1]$	$zX(z) - zx[0]$
18	Advance by two samples	$x[n+2]$	$z^2X(z) - z^2x[0] - zx[1]$
19	Advance by m samples	$x[n+m]$	$z^mX(z) - \sum_{k=0}^{m-1} x\,[k]z^{m-k}$
20	Exponential multiplication	$a^n x[n]$	$X(z/a)$
21	Multiplication by n	$nx[n]$	$-z\frac{dX}{dz}$
22	Running sum	$\sum_{k=0}^{n} x\,[k]$	$\frac{z}{z-1}X(z)$
23	Convolution	$\sum_{k=0}^{n} x\,[k]y[n-k]$	$X(z)Y(z)$
24	Periodic sequence period $N \geq 1$	$x[n+N] = x[n]$	$\frac{\sum_{k=0}^{N-1} x\,[k]z^{-k}}{1-z^{-N}}$

Conditions

- In shift formulas, m is a positive integer.
- Delays use zero extension for negative indices.
- Linearity and convolution hold in the common ROC; cancellations may enlarge it.
- Running sums converge at least where $X(z)$ converges and $|z| > 1$.
- For exponential multiplication, require z/a in the ROC of X.
- Differentiation holds inside the ROC.
- A nonzero periodic sequence has ROC $|z| > 1$.

Initial- and Final-Value Theorems

Theorem	Equation	Conditions
Initial Value	$x[0] = \lim_{z\to\infty} X(z)$	The one-sided transform converges for sufficiently large $\|z\|$.
Final Value	$\lim_{n\to\infty} x\,[n] = \lim_{z\to 1}(1-z^{-1})\,X(z)$	For rational $X(z)$, all poles of $(1-z^{-1})X(z)$, after cancellation, must lie strictly inside $\|z\| < 1$.

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See Also

- Harold's Partial Differential Equations (PDE) Cheat Sheet
- Harold's Differential Equation Models Cheat Sheet
- Harold's Euler's Method Example

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