Harold's Partial Differential Equations (PDE)
Cheat Sheet
26 September 2025

Notation

Notation	Expanded Form	Description
x_i x, y, t	$x_1, x_2, \ldots, x_n$ x, y, t	Variables
u	$u = u(x, y)$ $u = u(x, t)$	Function of two variables
u_x	$\dfrac{\partial u}{\partial x}$	Partial derivative of u with respect to x.
u_y	$\dfrac{\partial u}{\partial y}$	Partial derivative of u with respect to y.
u_t	$\dfrac{\partial u}{\partial t}$	Partial derivative of u with respect to t.
u_{xx}	$\dfrac{\partial^2 u}{\partial x^2}$	Partial second derivative of u with respect to x twice.
u_{yy}	$\dfrac{\partial^2 u}{\partial y^2}$	Partial second derivative of u with respect to y twice.
u_{xy}	$\dfrac{\partial^2 u}{\partial x \partial y}$	Partial second derivative of u with respect to x and y. $u_{xy} = u_{yx}$

Terminology

Term		Symbol	Definition
Ordinary Differential Equation (ODE)			A differential equation that involves an unknown function and its derivatives with respect to a single independent variable.
Partial Differential Equation (PDE)			A type of equation that involves an unknown function and its partial derivatives with respect to multiple independent variables.
Partial Derivative		$\dfrac{\partial u(x, y, t)}{\partial x}$	The derivative of a function with respect to one of its variables while treating all other variables as constants.
Linear			The unknown function and its derivatives appear to the first power and are not multiplied by each other.
Homogenous			There are no standalone terms (terms that do not contain the unknown function or its derivatives).
Linear Homogenous			All terms involve u and its derivatives.
Gradient (∇f)			$\nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}, \ldots \right) = \left(f_x, f_y, f_z, \ldots \right)$
Hessian ($\nabla^2 f$)			$H(f) = \begin{bmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{x_n x_1} & f_{x_n x_2} & \cdots & f_{x_n x_n} \end{bmatrix}$

First-Order PDEs

Description	Equation
General Form of Linear First-Order PDEs	$$F(x, y, u, u_x, u_y) = 0$$ $$F(x_i, u, \nabla u) = 0$$ $$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu + D = 0$$ where $u = u(x, y)$ and $A - D$ are functions.

Description	Name	Description
Classification	Linear	All coefficients are linear. A, B, C, D are $f(x, y)$.
	Semilinear	The coefficient of u can be nonlinear. A, B are $f(x, y)$, C is $f(x, y, u)$, D is 0.
	Quasilinear	Semi-linear + the coefficients of partial derivatives are functions of u as well. A, B, C are $f(x, y, u)$.
	Fully Nonlinear	One or all of the partial derivatives are nonlinear.

Notable First-Order PDEs

Description	Equation	Application
Transport Equation	$u_t + cu_x = 0$	Linear. Models the transport or advection of a quantity without dispersion or diffusion.
Burgers' Equation	$u_t + uu_x = 0$	Quasilinear. Describes the propagation of waves, often exhibiting shock waves.
Eikonal Equation	$(u_x)^2 + (u_x)^2 + (u_z)^2 = \dfrac{1}{v^2(x, y, z)}$	Nonlinear. Geometrical optics and other wave propagation problems, finding the shortest path in a medium.
Inviscid Burgers' Equation	$u_t + uu_x = 0$	Quasilinear. Fluid dynamics, traffic flow modeling, shock wave formation.
Hamilton-Jacobi-Bellman Equation	$\dfrac{\partial V}{\partial t} + \max_u \left\{ f(x, u) + \nabla V \cdot g(x, u) + \dfrac{1}{2} tr(\sigma(x, u)\sigma(x, u)^T \nabla^2 V) \right\} = 0$	
	Nonlinear. An alternative formulation of classical mechanics, equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics, and Hamiltonian mechanics.	

Second-Order PDEs

Description	Equation
General Form of Linear Second-Order PDEs	$F(x, y, u, u_{xx}, u_{xy}, u_{yy}, u_x, u_y) = G$ $A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + D\dfrac{\partial u}{\partial x} + E\dfrac{\partial u}{\partial y} + Fu = G$ where $u = u(x, y, t)$, $A - G$ are functions, and $G = 0$ is typical.

Description	$\Delta = B^2 - 4AC$	Discriminant	Applications
Classification	$\Delta < 0$	Elliptic	• Steady-state heat distribution • Electrostatic potential • Gravitational potential
	$\Delta = 0$	Parabolic	• Heat conduction • Diffusion processes • Option pricing in finance (Black-Scholes equation)
	$\Delta > 0$	Hyperbolic	• Sound waves • Electromagnetic waves • Traffic flow

Notable Second-Order PDEs

Description	Equation	Classification
1D Heat Equation (Diffusion)	$\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	Parabolic PDE
1D Wave Equation	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	Hyperbolic PDE
2D Heat Equation	$\dfrac{\partial u}{\partial t} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$	Parabolic PDE
2D Wave Equation	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$	Hyperbolic PDE
2D Poisson Equation	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = p(x, y)$	Elliptic PDE
2D Laplace Equation	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$	Elliptic PDE
3D Laplace Equation	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$	Elliptic PDE (Ellipsoid)
Tricomi Equation (Euler-Tricomi)	$\dfrac{\partial^2 u}{\partial x^2} + x\dfrac{\partial^2 u}{\partial y^2} = 0$	Elliptic for $y > 0$ Parabolic for $y = 0$ Hyperbolic for $y < 0$
2D Helmholtz Equation (Eigenvalue)	$-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = \lambda u$	Elliptic PDE
Schrödinger Equation	$-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + f(x, y)u = \iota\dfrac{\partial u}{\partial t}$	Elliptic PDE

Solution Approaches

Approach	Description
Analytical Methods	
Separation of Variables	This technique seeks solutions in the form of a product of functions, each dependent on a single independent variable (e.g., $u(x,t) = X(x)T(t)$ or $u(x,y) = X(x)Y(y)$. This transforms the PDE into a set of ordinary differential equations (ODEs) which are often easier to solve. This method is particularly effective for certain important PDEs like the heat equation and wave equation.
Fourier and Laplace Transforms	These integral transforms can be used to convert PDEs into algebraic equations, which are then solved in the transformed domain, and the solution is then transformed back to the original domain.
Green's Functions	This method utilizes Green's functions, which is a solution to the PDE with a point source term, to represent the solution to the original non-homogeneous PDE with given boundary conditions.
Method of Characteristics	This technique involves finding characteristic curves along which the PDE simplifies, which can lead to exact solutions for certain types of PDEs.
Numerical Methods	
Finite Difference Method (FDM)	This method approximates derivatives with finite differences, converting the PDE into a system of algebraic equations. FDM replaces partial derivatives with their approximations at discrete grid points within the domain.
Finite Element Method (FEM)	FEM divides the problem domain into smaller elements and approximates the solution within each element using basis functions. It then solves a system of equations derived from applying the PDE to each element and assembling the results for the entire domain. FEM is widely used for solving PDEs and has numerous variants like the hp-FEM.
Finite Volume Method (FVM)	Similar to FEM, FVM discretizes the domain into control volumes and applies conservation principles to each volume.
Method of Lines (MOL)	This technique discretizes all spatial dimensions of a PDE, leaving the time variable continuous, resulting in a system of ODEs. These ODEs can then be solved using standard ODE solvers.

Sources

- See also [Harold's Differential Equations Cheat Sheet](#)
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