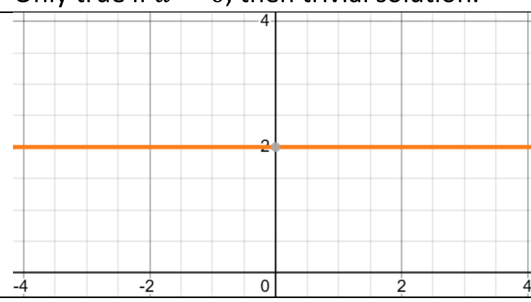
Harold's Polynomials
Cheat Sheet
14 May 2026

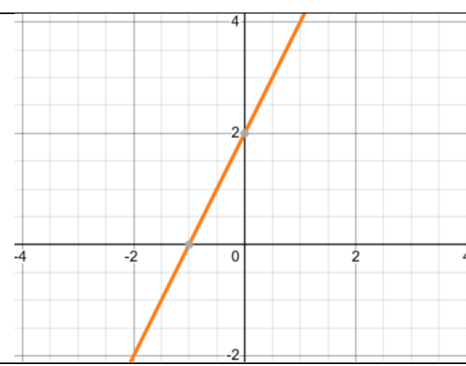
Polynomials

Description	Name	Equation	Max. Roots
0th Degree	Constant	$y = a$	0
1st Degree	Linear	$ax + b = 0$	1
2nd Degree	Quadratic	$ax^2 + bx + c = 0$	2
3rd Degree	Cubic	$ax^3 + bx^2 + cx + d = 0$	3
4th Degree	Quartic	$ax^4 + bx^3 + cx^2 + dx + e = 0$	4
5th Degree	Qunitic	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$	5
nth Degree	$P_n(x)$	$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$	n

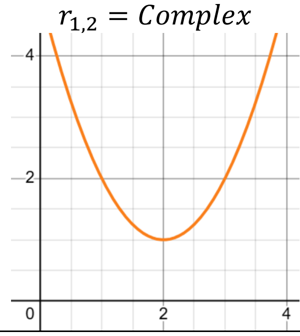
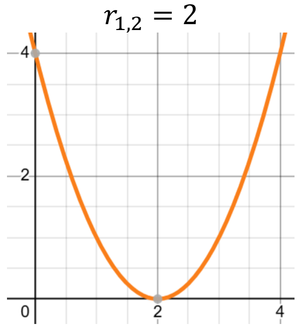
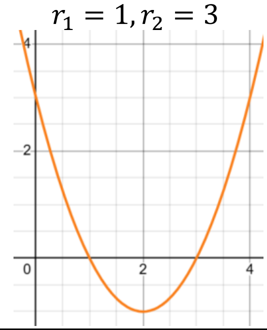
0. Constant Formula (a)

Description	Equation	Notes
Constant Formula Equation	$y = a$	
Solution	No roots	Only true if $a = 0$, then trivial solution.
Graph	No Real roots $y = 2$ $r_1 = DNE$	

1. Linear Formula (x)

Description	Equation	Notes
Linear Formula Equation	$ax + b = 0$	a and b can be negative.
Solution	$x = -\frac{b}{a}$	Simple algebra solution
Graph	1 Real root $y = 2x + 2$ $r_1 = -1$	

2. Quadratic Formula (x^2)

<table>
<tr><th>Description</th><th colspan="3">Equation</th><th>Notes</th></tr>
<tr><td>Quadratic Formula Equation</td><td colspan="3">$ax^2 + bx + c = 0$</td><td>Remember, b and c can be negative.</td></tr>
<tr><td>Quadratic Formula
(zeros, roots, x-intercepts)</td><td colspan="3">$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$</td><td>Discriminant:
• $b^2 - 4ac > 0$: two real unequal roots
• $b^2 - 4ac = 0$: repeated real roots (multiplicity of two)
• $b^2 - 4ac < 0$: two complex roots</td></tr>
<tr><td>Completing the Square</td><td colspan="4">1. Start with the quadratic formula form
$$ax^2 + bx + c = 0$$
2. Divide both sides by a
$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$$
3. Subtract the constant on the left to move it to the right side
$$x^2 + \left(\frac{b}{a}\right)x = -\left(\frac{c}{a}\right)$$
4. Cut the middle term in half, square it, then add it to both sides
$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}} = -\left(\frac{c}{a}\right) + \left(\frac{\mathbf{b}}{\mathbf{2a}}\right)^{\mathbf{2}}$$
5. Use an identity to factor the left side
$$a^2 + 2ab + b^2 = (a + b)^2$$
$$\left[x + \left(\frac{b}{2a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 - \left(\frac{c}{a}\right)\left(\frac{\mathbf{4a}}{\mathbf{4a}}\right)$$
6. Take the square root of both sides
$$x + \left(\frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$
7. Solve for x by subtracting the constant on the left to move it to the right
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$</td></tr>
<tr><td rowspan="2">Vertex</td><td colspan="4">Vertex: $x = -b/2a$
Finds the minimum / maximum of a parabola.</td></tr>
<tr><td colspan="4">$\frac{d}{dx}[cx^n] = cnx^{n-1} = 0$
Calculus Power Rule method</td></tr>
<tr><td>Graphs</td><td>0 Real roots
$y = (x-2)^2 + 1$
$r_{1,2} = Complex$
</td><td>2 Real same roots
$y = (x-2)^2$
$r_{1,2} = 2$
</td><td colspan="2">2 Real different roots
$y = (x-2)^2 - 1$
$r_1 = 1, r_2 = 3$
</td></tr>
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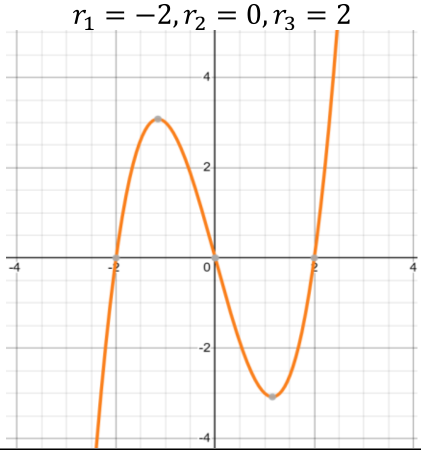
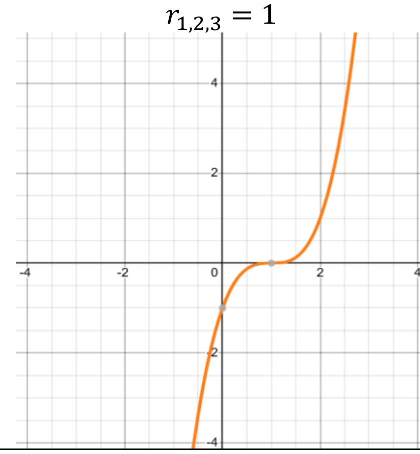
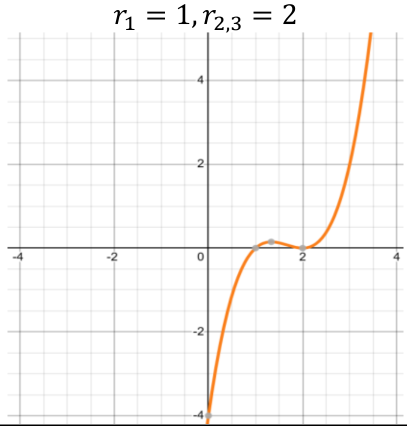
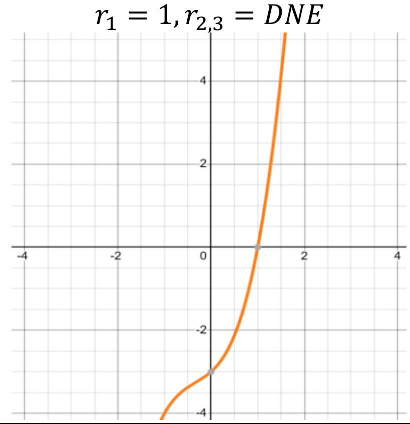
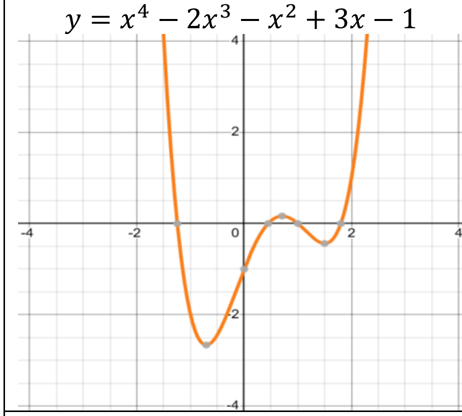
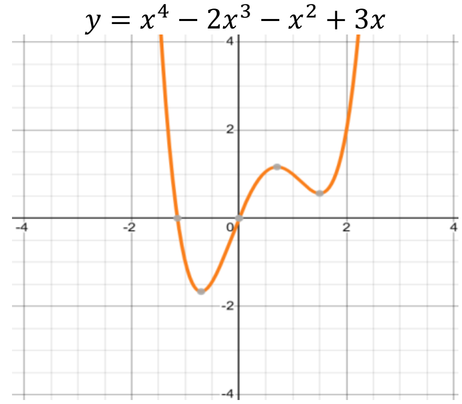
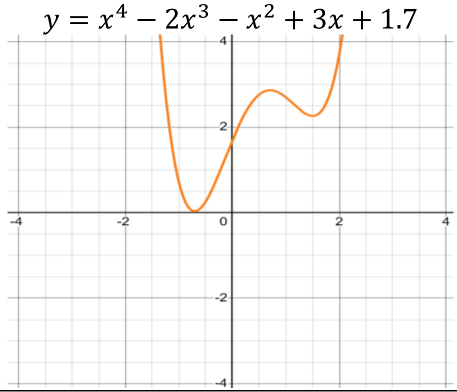
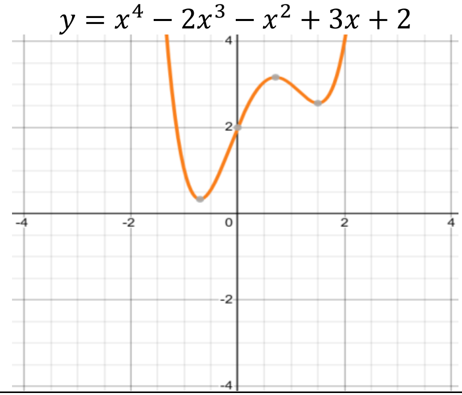
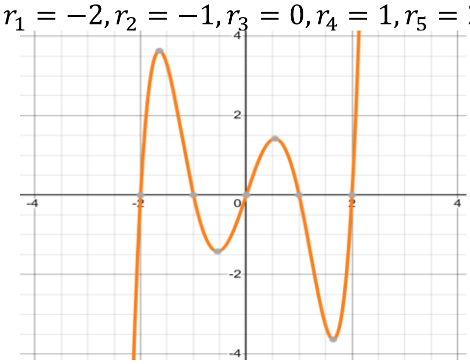
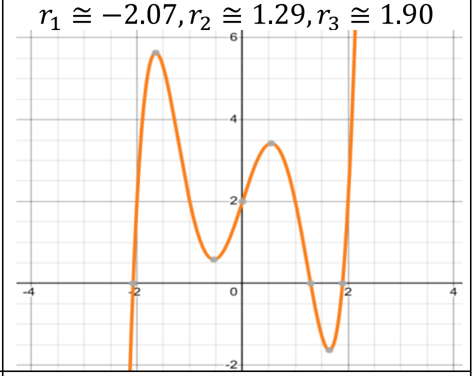
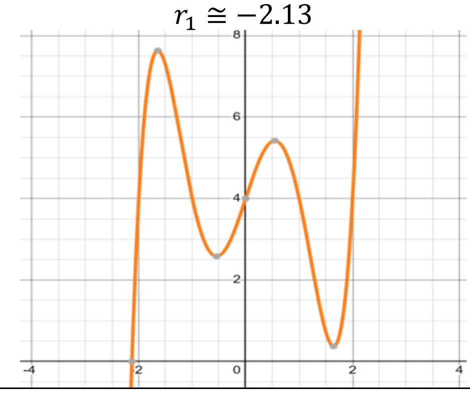
3. Cubic Formula (x^3)

Description	Equation
Cubic Formula Equation	$ax^3 + bx^2 + cx + d = 0$
Solution	$\Delta_0 = b^2 - 3ac$ $\Delta_1 = 2b^3 - 9abc + 27a^2d$ $C = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$ $r_1 = -\frac{1}{3a}\left(b + C + \frac{\Delta_0}{C}\right)$ $r_{2,3} = -\frac{1}{3a}\left(b + \frac{-1 \pm i\sqrt{3}}{2}C + \frac{-1 \mp i\sqrt{3}}{2}\frac{\Delta_0}{C}\right)$
Graphs	3 Real different roots $y = x^3 - 4x$ $r_1 = -2, r_2 = 0, r_3 = 2$ 
	3 Real same roots $y = x^3 - 3x^2 + 3x - 1$ $r_{1,2,3} = 1$ 
	2 Real different roots $y = x^3 - 5x^2 + 8x - 4$ $r_1 = 1, r_{2,3} = 2$ 
	1 Real root $y = x^3 + x^2 + x - 3$ $r_1 = 1, r_{2,3} = DNE$ 

4. Quartic Formula (x^4)

Description	Equation	
Quartic Formula Equation	$ax^4 + bx^3 + cx^2 + dx + e = 0$	
Solution	$\Delta_0 = c^2 - 3bd + 12ae$ $\Delta_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2c - 72ace$ $p = \dfrac{8ac - 3b^2}{8a^2} \qquad q = \dfrac{b^3 - 4abc + 8a^2d}{8a^3}$ $Q = \sqrt[3]{\dfrac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \qquad S = \dfrac{1}{2}\sqrt{-\dfrac{2}{3}p + \dfrac{1}{3a}\left(Q + \dfrac{\Delta_0}{Q}\right)}$ $r_{1,2} = -\dfrac{b}{4a} - S \pm \dfrac{1}{2}\sqrt{-4S^2 - 2p + \dfrac{q}{S}}$ $r_{3,4} = -\dfrac{b}{4a} + S \pm \dfrac{1}{2}\sqrt{-4S^2 - 2p + \dfrac{q}{S}}$	
Graphs	4 Real different roots $y = x^4 - 2x^3 - x^2 + 3x - 1$ 	2 Real different roots $y = x^4 - 2x^3 - x^2 + 3x$ 
	2 Real same roots $y = x^4 - 2x^3 - x^2 + 3x + 1.7$ 	0 Real roots $y = x^4 - 2x^3 - x^2 + 3x + 2$ 

5. Quintic Formula (x^5)

Description	Equation	
Quintic Formula Form	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$	
Solution	No general formula exists with arbitrary coefficients. Abel-Ruffini Theorem: No general solution exists for degree 5 or higher.	
Graphs	5 Real different roots $y = x^5 - 5x^3 + 4x$ $r_1 = -2, r_2 = -1, r_3 = 0, r_4 = 1, r_5 = 2$ 	3 Real different roots $y = x^5 - 5x^3 + 4x + 2$ $r_1 \cong -2.07, r_2 \cong 1.29, r_3 \cong 1.90$ 
	1 Real root $y = x^5 - 5x^3 + 4x + 4$ $r_1 \cong -2.13$ 	

Polynomial Division

<table>
<tr><th>Description</th><th>Equations</th></tr>
<tr><td>Long Division</td><td>

$$f(x) = \frac{x^3 - 2x^2 - 4}{x - 3}$$

$$\begin{array}{rl} & x^2 + \ \ x + 3 \\ x - 3 \ ) & \overline{x^3 - 2x^2 + 0x - 4} \\ & \underline{x^3 - 3x^2} \\ & \quad +x^2 + 0x \\ & \quad \underline{+x^2 - 3x} \\ & \qquad +3x - 4 \\ & \qquad \underline{+3x - 9} \\ & \qquad\qquad +5 \end{array}$$

$$\frac{x^3 - 2x^2 - 4}{x - 3} = \underbrace{x^2 + x + 3}_{q(x)} + \frac{\overbrace{5}^{r(x)}}{x - 3}$$

</td></tr>
<tr><td>Synthetic Division</td><td>

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

x_0	a_n	a_{n-1} …	a_1	a_0
	↓	$a_n x_0$	$x_0(a_{n-1} + a_n x_0)$	*(mult.)*
	a_n	$a_{n-1} + a_n x_0$	*(add)*	$f(x_0)$

- Synthetic division is faster than long division.
- The top row lists the polynomial coefficients in descending order.
- If a term is missing, add a 0 as a placeholder.
- x_0 is a root candidate.
- Add columns.
- Multiple bottom terms by x_0 to get the next entry to add.
- **Remainder Theorem**: If 0 comes out of the bottom right, meaning $f(x_0) = 0$, then x_0 is a factor, and the bottom row is also a polynomial factor of degree $n - 1$.

Example:

$$P(x) = 2x^3 - 5x^2 + x - 6$$

2	**2**	**-5**	**1**	**-6**
	↓	4	2	6
	2	-1	3	0

So, $x = 2$ is a root, and $(x - 2)$ is a factor of $P(x)$.

$$P(x) = 2x^3 - 5x^2 + x - 6 = (x - 2)(2x^2 - x + 3)$$

</td></tr>
<tr><td>Remainder Theorem</td><td>

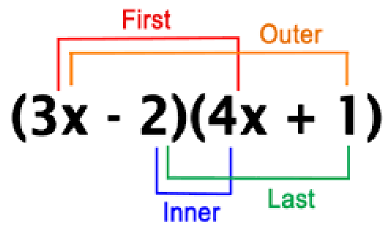
1) If a polynomial $P(x)$ is divided by $(x - k)$, then the remainder is $P(k)$.
2) $P(a) = 0$ if and only if $(x - a)$ is a factor of polynomial $P(x)$.

</td></tr>
</table>

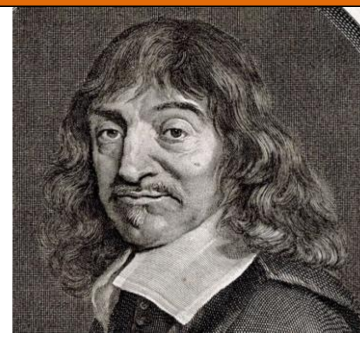
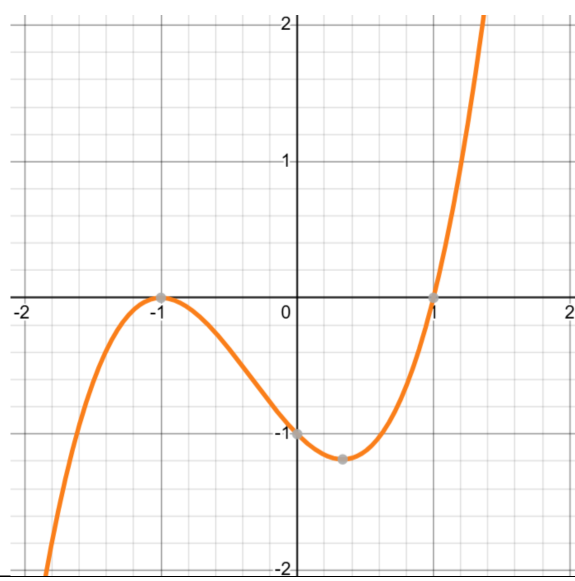
Rational Root Theorem

Description	Equations
Rational Root Theorem	Tells you all *possible* rational roots of a polynomial with integer coefficients. It doesn't guarantee which ones actually work — it just gives the complete **list to test**.
Prerequisites	• Polynomial with integer coefficients.
Results	• The set of rational root candidates in the form $\frac{p}{q}$. • A finite list to test using synthetic division.
Polynomial Equation	$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$
Formula	$x = \frac{p}{q} = \pm\frac{factors\ of\ a_0}{factors\ of\ a_n}$
Example	$P(x) = 2x^3 - 5x^2 + x - 6$ • Constant term: $a_0 = -6$ → factors: $1, 2, 3, 6$ • Leading coefficient: $a_n = 2$ → factors: $1, 2$ Possible rational zeros: $\pm\mathbf{1}, \pm\mathbf{2}, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{\mathbf{3}}{\mathbf{2}}, \pm\frac{6}{2} = \pm 3$ Synthetic division: $\begin{array}{c\|cccc} \mathbf{2} & \mathbf{2} & \mathbf{-5} & \mathbf{1} & \mathbf{-6} \\ & \downarrow & 4 & 2 & 6 \\ \hline & 2 & -1 & 3 & 0 \end{array}$ Solution: $P(x) = (x-2)(2x^2 - x - 3)$ $= (x-2)(2x-3)(x+1)$

Factoring Formulas

Description	Equation	Notes
Quadratic	$x^2 + a^2 = (x + ai)\,(x - ai)$ $x^2 - a^2 = (x + a)\,(x - a)$ $(x + a)^2 = x^2 + 2ax + a^2$ $(x - a)^2 = x^2 - 2ax + a^2$ $(x + a)\,(x + b) = x^2 + (a + b)x + ab$ $(x - a)\,(x - b) = x^2 - (a + b)x + ab$	FOIL: 
Cubic	$x^3 + a^3 = (x + a)\,(x^2 - ax + a^2)$ $x^3 - a^3 = (x - a)\,(x^2 + ax + a^2)$ $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$ $(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$	Notice one '–' and two '+' for both Binomial expansion (Pascal's Triangle)
Even Powers	$x^{2n} + a^{2n} = (a^n - ix^n)\,(a^n + ix^n)$ $x^{2n} - a^{2n} = (x^n - a^n)\,(x^n + a^n)$	The top equation is not often used since it requires imaginary numbers (i)
Odd Powers	$x^n + a^n = (x + a)\,(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots - a^{n-2}x + a^{n-1})$ $x^n - a^n = (x - a)\,(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$	

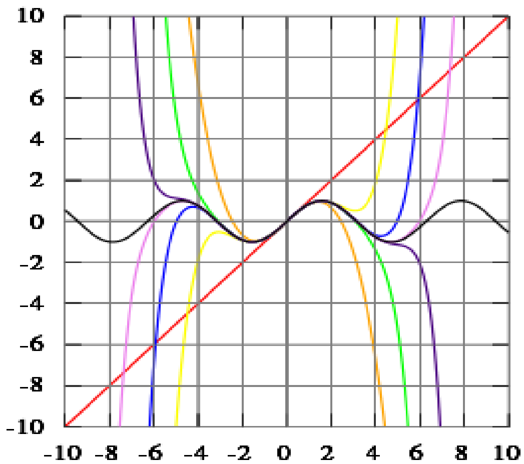
Descartes' Rule of Signs

Description	Equations
Descartes' Rule of Signs	Counts the roots of a polynomial by examining sign changes in its coefficients.  René Descartes (1596-1650)
Prerequisites	• Single-variable polynomial (*x*) with real coefficients. • Ordered by descending variable exponent (height to low). • Omit zero coefficients.
Results	• Finds real roots only. • Finds the number of possible positive and negative roots. • A root of multiplicity *k* is counted as *k* roots.
Positive Roots	The number of <u>positive</u> roots of the polynomial is either: 1) Equal to the number of sign changes between consecutive (nonzero) coefficients, or is 2) Less than it by an even number
Negative Roots	The number of <u>negative</u> roots of the polynomial is either 1) Equal to the number of sign changes **after multiplying the coefficients of odd-power terms by −1**, or is 2) Fewer than it by an even number
Example	$P_3(x) = x^3 + x^2 - x - 1$ $(+\ +\ -\ -)$ → One sign change, so exactly 1 <u>positive</u> root. $(-\ +\ +\ -)$ → Two sign changes, so 2 or 0 <u>negative</u> roots. Actual roots: −1, −1, 1 

Vieta's Formula

Description	Equations		
Vieta's Formula	Provides a direct connection between a polynomial's coefficients and its roots. $\sum_{1 \le i_1 < i_2 < \ldots < i_k < n} \left(\prod_{j=1}^{k} r_{i_j} \right) = (-1)^k \frac{a_{n-k}}{a_n}$	 François Viète (1540-1603) Franciscus Vieta (Latin)	
Prerequisites	• Coefficients can be real or complex numbers. • $a_0 \neq 0$		
Polynomial Equations	$P_2(x) = ax^2 + bx + c$ $P_3(x) = ax^3 + bx^2 + cx + d$ $P_4(x) = ax^4 + bx^3 + cx^2 + dx + e$ $P_5(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$		
	Sum of Roots	**Sum of Products of Roots**	**Product of Roots**
General Formulas	$\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n}$	Pairs of two: $\sum_{i,j=1,i\neq j}^{n} r_i r_j = \frac{a_{n-2}}{a_n}$ Pairs of three: $\sum_{i,j,k=1,i\neq j\neq k}^{n} r_i r_j r_k = -\frac{a_{n-3}}{a_n}$	$r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}$
2. Quadratic	$r_1 + r_2 = -\frac{b}{a}$	(same as Product of Roots)	$r_1 r_2 = \frac{c}{a}$
3. Cubic	$r_1 + r_2 + r_3 = -\frac{b}{a}$	$r_1r_2 + r_1r_3 + r_2r_3 = \frac{c}{a}$	$r_1r_2r_3 = -\frac{d}{a}$
4. Quartic	$r_1 + r_2 + r_3 + r_4 = -\frac{b}{a}$	$r_1r_2 + \ldots + r_3r_4 = \frac{c}{a}$ $r_1r_2r_3 + \ldots + r_2r_3r_4 = -\frac{d}{a}$	$r_1r_2r_3r_4 = \frac{e}{a}$
5. Quintic	$r_1 + r_2 + r_3 + r_4 + r_5 = -\frac{a_4}{a_5}$ $r_1r_2 + r_1r_3 + \ldots + r_4r_5 = \frac{a_3}{a_5}$ $r_1r_2r_3 + \ldots + r_3r_4r_5 = -\frac{a_2}{a_5}$ $r_1r_2r_3r_4 + \ldots + r_2r_3r_4r_5 = \frac{a_1}{a_5}$ $r_1r_2r_3r_4r_5 = -\frac{a_0}{a_5}$		

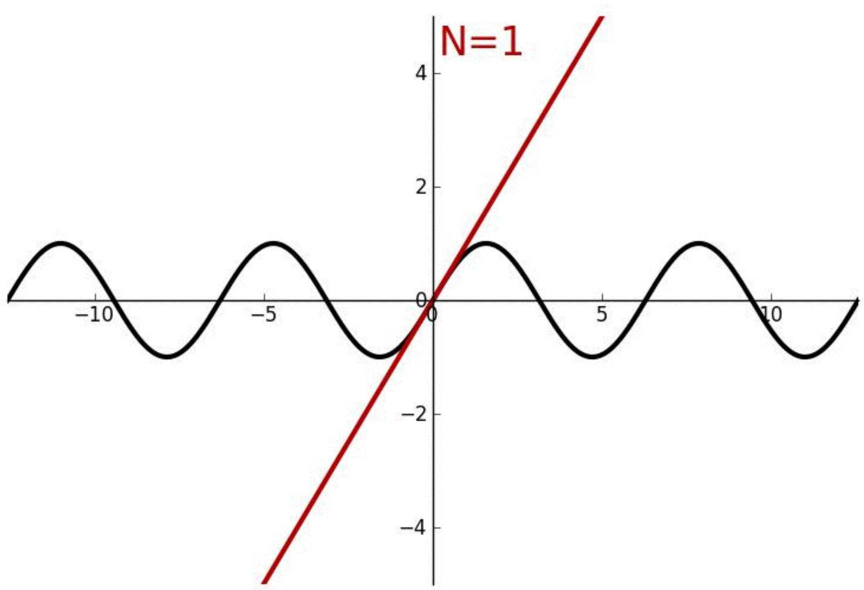
Polynomial Approximation Using Calculus

Approximation Polynomial	
	$\boldsymbol{f(x) = P_n(x) + R_n(x)}$ $P_n(x) = n^{th}$ degree polynomial approximation $R_n(x) = n^{th}$ degree polynomial remainder $R_n(x) = \pm\, Error$ NOTE: $P_n(x)$ is easy to integrate and differentiate

Maclaurin Series	
Maclaurin Series Taylor Series centered about $x = 0$	$\boldsymbol{f(x) \approx P_n(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!}\, x^n}$
Maclaurin Series Remainder	$R_n(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!}\, x^{n+1}$ where $x \leq x^* \leq max$ and $\lim_{x \to +\infty} R_n(x) = 0$ NOTE : x^* is the worst-case scenario x for this interval.

Taylor Series	
Taylor Series Maclaurin Series if $c = 0$	$f(x) \approx P_n(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}\, (x-c)^n$
Taylor Series Remainder	$R_n(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!}\, (x-c)^{n+1}$ where $x \leq x^* \leq c$ and $\lim_{x \to +\infty} R_n(x) = 0$ NOTE : x^* is the worst-case scenario x for this interval.
 Brook Taylor (1685-1731)	 Colin Maclaurin (1698-1746)

Common Polynomial Approximations	Expanded Form
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$
$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ for $\lvert x\rvert < 1$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \cdots$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $\lvert x\rvert < 1$	$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots$
$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all x	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots$
$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for all x	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \cdots$
$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$ for $\lvert x\rvert < 1$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \; for -1 < x < 1$ $\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \cdots \; for\ x \geq 1$ $-\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \cdots \; for\ x < 1$
$\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ for all x	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + \cdots$
$\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ for all x	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \frac{x^{14}}{14!} + \cdots$



See

- Harold’s Infinite Series Cheat Sheet
- Harold’s Taylor Series Cheat Sheet

Sources

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