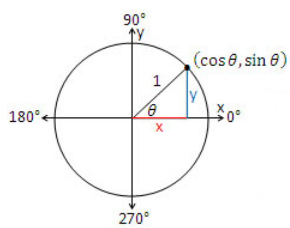
Harold's Trig Proofs
Cheat Sheet
15 March 2023

After proving only three (3) trig formulas, we can easily derive **ALL** the trig formulas!

1. Pythagorean Identity
2. Sum and difference formula for sine
3. Sum and difference formula for cosine

Proof of Pythagorean Identities

Proof

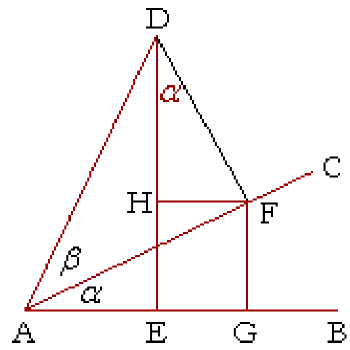
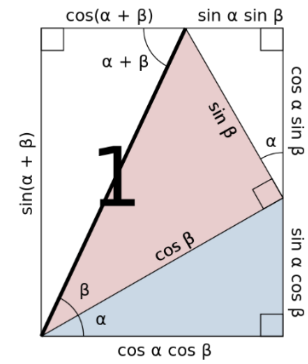
Given 	Pythagorean Theorem $x^2 + y^2 = r^2$ $r = 1$ $\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} = \frac{y}{1} = y$ $\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} = \frac{x}{1} = x$
Substitute and Simplify	$\sin^2\theta + \cos^2\theta = 1^2$
Formula	$\mathbf{\sin^2\theta + \cos^2\theta = 1 \quad [1]}$

Proof

Given	$\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Divide by $\cos^2\theta$, then Simplify	$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$
Formula	$\mathbf{\tan^2\theta + 1 = \sec^2\theta \quad [2]}$

Proof

Given	$\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Divide by $\sin^2\theta$, then Simplify	$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$
Formula	$\mathbf{1 + \cot^2\theta = \csc^2\theta \quad [3]}$

Proof of Sum and Difference Formulas	
Trig Sum and Difference Formulas	$\boldsymbol{\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta}$ $\boldsymbol{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$
Proof Diagram	 

Proof of *sin*$(\alpha \pm \beta)$	
Prove Sum	
Given	$\sin(\alpha+\beta) = \frac{ED}{DA} = \frac{opposite}{hypotenuse}$
Alternate interior angles are congruent	$\alpha = \angle CAB = \angle HFA = \angle HDF$
Tallest vertical line	$ED = GF + HD$
Substitute, then divide and multiply by AF & FD	$\sin(\alpha+\beta) = \frac{ED}{AD} = \frac{GF}{AD} + \frac{HD}{AD} = \frac{GF}{AF}\frac{AF}{AD} + \frac{HD}{FD}\frac{FD}{AD}$
Convert back to trig formulas	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad [4]$
Prove Difference	
Replace $+\beta$ with $-\beta$	$\cos(-\beta) = \cos(\beta)$ $\sin(-\beta) = -\sin(\beta)$
Simplify	$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad [5]$
General Formula [4+5]	$\boldsymbol{\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad [6]}$

Proof of $cos(\alpha \pm \beta)$	
Prove Sum	
Given	$\cos(\alpha+\beta) = \frac{AE}{AD} = \frac{adjacent}{hypotenuse}$
Longest horizontal line	$EA = GA - FH$
Substitute, then divide and multiply by AF & DF	$\cos(\alpha+\beta) = \frac{EA}{AD} = \frac{GA}{AD} - \frac{FH}{AD} = \frac{GA}{AF}\frac{AF}{AD} + \frac{FH}{DF}\frac{DF}{AD}$
Convert back to trig formulas	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad [7]$
Prove Difference	
Replace $+\beta$ with $-\beta$	$\cos(-\beta) = \cos(\beta)$ $\sin(-\beta) = -\sin(\beta)$
Simplify	$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad [8]$
General Formula [7+8]	$\mathbf{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \quad [9]}$

Proof of $tan(\alpha \pm \beta)$	
Prove Sum and Difference	
Given	$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)}$
Substitute	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad [6]$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \quad [9]$
Divide by $(\cos\alpha\cos\beta)$, then Simplify	$\tan(\alpha \pm \beta) = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$
General Formula	$\mathbf{\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\ \tan\beta} \quad [10]}$

Proof of Double Angle Formulas (2θ)

Proof

Given	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad [4]$
Substitute	$\theta = \alpha = \beta$
Simplify	$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$
Formula	$\boldsymbol{\sin(2\theta) = 2\sin\theta\cos\theta \quad [14]}$

Proof

Given	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad [7]$
Substitute	$\theta = \alpha = \beta$
Simplify	$\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$
Formula	$\boldsymbol{\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]}$

Proof

Given	$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]$ $\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Substitute	$\sin^2\theta = 1 - \cos^2\theta$
Simplify	$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta)$
Formula	$\boldsymbol{\cos(2\theta) = 2\cos^2\theta - 1 \quad [16]}$

Proof

Given	$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]$ $\sin^2\theta + \cos^2\theta = 1 \quad [1]$
Substitute	$\cos^2\theta = 1 - \sin^2\theta$
Simplify	$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta$
Formula	$\boldsymbol{\cos(2\theta) = 1 - 2\sin^2\theta \quad [17]}$

Proof

Given	$\tan(2\theta) = \dfrac{\sin(2\theta)}{\cos(2\theta)}$
Substitute	$\sin(2\theta) = 2\sin\theta\cos\theta \quad [14]$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad [15]$
Divide by $\cos^2\theta$	$\tan(2\theta) = \dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$
Simplify	$\tan(2\theta) = \dfrac{\left(\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}\right)}{\left(\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}\right)}$
Formula	$\boldsymbol{\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta} \quad [18]}$

Proof of Half Angle Formulas (θ/2)

Proof

Given	$\cos(2\theta) = 1 - 2\sin^2\theta \quad [17]$
Solve for $\sin^2\theta$	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2} \quad [19a]$
Substitute	$\theta = \dfrac{\theta}{2}$
Solve	$\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos(\theta)}{2}$
Formula	$\mathbf{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}} \quad [\mathbf{19}\boldsymbol{b}]$

Proof

Given	$\cos(2\theta) = 2\cos^2\theta - 1 \quad [16]$
Solve for $\cos^2\theta$	$\cos^2\theta = \dfrac{1+\cos(2\theta)}{2} \quad [20a]$
Substitute	$\theta = \dfrac{\theta}{2}$
Solve	$\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1+\cos(\theta)}{2}$
Formula	$\mathbf{\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}} \quad [\mathbf{20}\boldsymbol{b}]$

Proof

Given	$\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$
Substitute	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2} \quad [19a]$ $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2} \quad [20a]$
Simplify	$\tan^2\theta = \dfrac{\left(\frac{1-\cos(2\theta)}{2}\right)}{\left(\frac{1+\cos(2\theta)}{2}\right)} = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Substitute	$\theta = \dfrac{\theta}{2}$
Solve	$\tan^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos(\theta)}{1+\cos(\theta)} \quad [21a]$
Formula	$\mathbf{\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}} \quad [\mathbf{21}\boldsymbol{b}]$

Proof of Cofunction Formulas

Proof

Given	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \quad [5]$
Substitute	$\alpha=\frac{\pi}{2}, \beta=\theta$
Simplify	$\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)\cos\theta+\cos\left(\frac{\pi}{2}\right)\sin\theta$
Formula	$\mathbf{\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad [22]}$

Proof

Given	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \quad [8]$
Substitute	$\alpha=\frac{\pi}{2}, \beta=\theta$
Simplify	$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}\right)\cos\theta-\sin\left(\frac{\pi}{2}\right)\sin\theta$
Formula	$\mathbf{\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad [23]}$

Proof

Given	$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$
Substitute	$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad [22]$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad [23]$
Simplify	$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{\cos\theta}{\sin\theta}=\cot\theta$
Formula	$\mathbf{\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta \quad [24]}$

Proof

Given	$\sec(\theta)=\frac{1}{\cos(\theta)}$
Substitute	$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad [23]$
Simplify	$\sec\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)}=\frac{1}{\sin(\theta)}=\csc\theta$
Formula	$\mathbf{\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \quad [25]}$

Proof

Given	$\csc(\theta) = \dfrac{1}{\sin(\theta)}$
Substitute	$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta \quad [22]$
Simplify	$\csc\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\sin\left(\frac{\pi}{2} - \theta\right)} = \dfrac{1}{\cos(\theta)} = \sec\theta$
Formula	$\boldsymbol{\csc\left(\dfrac{\pi}{2} - \theta\right) = \sec\theta \quad [26]}$

Proof

Given	$\cot(\theta) = \dfrac{1}{\tan(\theta)}$
Substitute	$\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta \quad [24]$
Simplify	$\cot\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\left(\frac{\pi}{2} - \theta\right)} = \dfrac{1}{\cot(\theta)} = \tan\theta$
Formula	$\boldsymbol{\cot\left(\dfrac{\pi}{2} - \theta\right) = \tan\theta \quad [27]}$

Additional Resources

- AoPSOnline (2025). Art of Problem Solving, Proofs of trig identities. https://artofproblemsolving.com/wiki/index.php/Proofs_of_trig_identities